The local Gan–Gross–Prasad conjecture for general spin groups: a Lean blueprint

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How to read this blueprint

This blueprint accompanies the Lean 4 formalization GSpinGGP (work in progress) of the dissertation [1]: S. Pease, *The Local Gan–Gross–Prasad Conjecture for General Spin Groups*, Rutgers University–Newark, 2026, doi:10.7282/t3-qb3s-s249. It is cited as "Thesis", with the numbers of the printed dissertation. The proof adapts Beuzart-Plessis's local trace formula for unitary groups [5] and the work of Waldspurger [26] and Luo [17] on special orthogonal groups.

Every node is a Lean declaration, and every statement is formalized. The main theorem (Theorem 60) is proved in Lean from three kinds of inputs: results whose Lean proof is still missing (`sorry`), results from the literature that are assumed (Lean `axiom`s), and definitions. Results are attributed as follows.

- **Proved**: the Lean proof of this step contains no `sorry`. It may still rely, through the results it uses, on unproved steps or on the literature.
- **Not yet proved**: the Lean proof is `sorry`. Its proof paragraph gives the intended argument.
- **Assumed from the literature**: a "Literature" environment, with an "Assumed from" line giving the source. These are Lean `axiom`s; they have no proof paragraph, and are never proved in this project. Each is a published statement, specialized to the groups and fields used here, with the source's hypotheses checked; results the sources prove only for other groups (unitary, special orthogonal, …) and that are adapted here are stated as theorems instead (proved, or left open with a proof sketch), and their proof paragraphs list the ingredients.
- **Definitions**: real Lean definitions (never axiomatized).

Where a Lean statement deliberately differs from the dissertation, its node says so in one line.

Colours in the dependency graph. Boxes are definitions, ellipses are results. A green border means the statement is formalized, which is the case for every node. The interior colour gives the status of the proof:

- *light green fill*: proved (the Lean proof of this step has no `sorry`);
- *dark green fill*: proved, and so is everything it depends on (no `sorry` and no literature assumption among its ancestors);
- *blue fill*: not yet proved (`sorry`); the statement and everything it uses are formalized, so the proof is ready to be written;

- *no fill (white)*, with a label starting with `LIT-`: assumed from the literature (Lean `axiom`);
- *light green box*: a definition.

Because every result on the path of the main theorem rests on literature assumptions, none of them is dark green: the dark green nodes are the self-contained results. The graph's own legend uses the generic wording of the blueprint software ("ready to be formalized" for the blue fill).

Status (November 2026). The project builds with 45 declarations containing `sorry`; the main theorem depends on 33 of them and on 11 of the 12 literature assumptions. Classical facts about quadratic forms (Witt cancellation and decomposition, Cartan–Dieudonné, the classification over p-adic fields) are proved from the Tau Ceti library; Hilbert's Theorem 90 for GL_m is proved from Mathlib. Only results stated verbatim in the literature (specialized to general spin groups, with the source's hypotheses checked) are literature assumptions; where the dissertation adapts or combines published results, the statement is a theorem whose proof is not yet formalized, and its proof sketch names the exact ingredients. The 11 unproved statements not on its path are collected in Chapter 9. To see exactly what the main theorem rests on, run `#print axioms GSpinGGP.local_ggp`.

Chapter 1

General spin groups and GGP triples

This chapter follows Thesis Ch. 5. Throughout, F is a p-adic field (a finite extension of $\mathbb{Q}_p$) and ψ a nontrivial continuous additive character of F. Quadratic spaces are nondegenerate and finite-dimensional; b is the bilinear form with $q(v) = b(v, v)$.

1.1 Quadratic spaces and the general spin group

Definition 1 (The general spin group)**.** For a quadratic space (V, q) with Clifford algebra $C(V)$ and even part $C^+(V)$, the general spin group is

$$\mathrm{GSpin}(V) = \{g \in C^+(V)^\times : gVg^{-1} = V\}$$

(Thesis §5.1.2). Conjugation gives a surjection $P : \mathrm{GSpin}(V) \to \mathrm{SO}(V)$ on $\overline{F}$-points with kernel the scalars $\mathbb{G}_m$. (This is not the spin group of unit spinor norm.)

Definition 2 (The Lie algebra $\mathfrak{gspin}(V)$)**.** $\mathfrak{gspin}(V) = \{X \in C^+(V) : [X, v] \in V \text{ for all } v \in V\}$, a Lie subalgebra of $C(V)$ for the commutator bracket (Thesis Prop. 5.1.3.1); $\mathfrak{gspin}(V) = \mathfrak{so}(V) \oplus F$.

Theorem 3 (Witt decomposition)**.** *A quadratic space is $V \cong H^w \perp V_{\mathrm{an}}$ with H the hyperbolic plane and V_{an} anisotropic, where $w = w(V)$ is the Witt index (the maximal dimension of a totally isotropic subspace). In particular $\dim V_{\mathrm{an}} = \dim V - 2w(V)$, so $\mathrm{GSpin}(V)$ is quasi-split iff $\dim V - 2w(V) \le 2$.*

Proof. Proved from Tau Ceti's Witt decomposition of a regular form ([16], Ch. I §4). □

Theorem 4 (Witt cancellation)**.** *Over a field of characteristic $\neq 2$, if $U \perp X \cong U \perp Y$ then $X \cong Y$ (all forms finite-dimensional, possibly degenerate).*

Proof. Split off the radicals and apply Tau Ceti's Witt cancellation for a regular summand ([16], Ch. I §4). □

Theorem 5 (Quadratic forms over a p-adic field)**.** *Let Q be a nondegenerate quadratic form on F^m. The isometry classes of nondegenerate forms on F^m with the same discriminant as Q are*

finite in number, $n \le 2$ of them, and admit a system of representatives containing Q; $n = 2$ if $m \ge 3$, $n = 1$ if $m \le 1$, and for $m = 2$, $n = 1$ iff $-\det Q$ is a square.

Proof. Proved from Tau Ceti's classification of regular forms over a nonarchimedean local field by rank, discriminant and Hasse invariant, and its list of the realizable invariants ([16], Ch. VI §2; O'Meara §63), through the fact that a p-adic field is a nonarchimedean local field. □

Theorem 6 (Cartan–Dieudonné, SO form)**.** *In a nondegenerate quadratic space over a field of characteristic $\neq 2$, every isometry of determinant 1 is a product $\prod_i s_{u_i} s_{w_i}$ of pairs of reflections in anisotropic vectors.*

Proof. Proved from Tau Ceti's factorization of an element of SO into an even number of reflections ([4], Ch. III; [16], Ch. I), grouped in pairs. □

Theorem 7 (Centralizers of semisimple elements of GSpin)**.** *Let $x \in \mathrm{GSpin}(V)$ be semisimple, $\bar{x} = P(x)$, and $W_{\pm 1} = \ker(\bar{x} \mp 1)$. Then $Z_{\mathrm{GSpin}(V)}(x) = P^{-1}(Z_{\mathrm{SO}(V)}(\bar{x})^0)$; explicitly, g commutes with x iff $P(g)$ commutes with $\bar{x}$ and $\det P(g)|_{W_{+1}} = \det P(g)|_{W_{-1}} = 1$. (Characteristic 0.)*

Proof. Own derivation, with no literature input; it replaces the appeal to [22], Thm. 8.1. If $P(g)$ commutes with $\bar{x}$, then $gxg^{-1} = c\,x$ with $c = \pm 1$ (the commutator lies in $\ker P$, and the Clifford norm gives $c^2 = 1$). By base change to $\bar{K}$ it suffices to show $c = \det P(g)|_{W_{-1}}$ there (Lemma 8). The condition on W_{+1} then follows from $\det P(g) = \det P(g)|_{W_{+1}} \cdot \det P(g)|_{W_{-1}}$: the eigenspaces E_μ and $E_{\mu^{-1}}$ ($\mu \neq \pm 1$) are paired perfectly, so $P(g)$ has determinant 1 on their sum. □

Lemma 8 (The commutator sign)**.** *Over an algebraically closed field of characteristic 0, let $x \in \mathrm{GSpin}(V)$ with $\bar{x} = P(x)$ semisimple, and let $P(g)$ commute with $\bar{x}$. Then g commutes with x iff $\det P(g)|_{W_{-1}} = 1$.*

Proof. Cartan–Dieudonné on the nondegenerate spaces W_{+1} and W_{-1} writes $P(g)|_{W_{\pm 1}}$ as products of reflections in vectors $u_i \in W_{+1}$, $w_j \in W_{-1}$; their number is even, and $y = \prod u_i \prod w_j \in \mathrm{GSpin}(V)$. Since $xv = \pm vx$ for $v \in W_{\pm 1}$, $xy = (-1)^{\#\{w_j\}} yx$, and $(-1)^{\#\{w_j\}} = \det P(g)|_{W_{-1}}$. Finally $y^{-1}g$ commutes with x by Lemma 9. □

Lemma 9 (Pair blocks)**.** *Over an algebraically closed field of characteristic 0, let $\bar{x} = P(x)$ be semisimple, and let $P(g)$ commute with $\bar{x}$ and be the identity on $W_{+1} \oplus W_{-1}$. Then g commutes with x.*

Proof. Induction over the eigenvalues $\mu \neq \pm 1$ of $\bar{x}$. $P(g)$ preserves E_μ; by Lemma 10 some γ commuting with x agrees with $P(g)$ on E_μ and is the identity on all E_ν, $\nu \neq \mu, \mu^{-1}$. Then $P(\gamma^{-1}g)$ is the identity on E_μ, hence on $E_{\mu^{-1}}$ (the two are perfectly paired and $P(\gamma^{-1}g)$ is an isometry). At the end $P(g') = 1$, so g' is a scalar. □

Lemma 10 (Rank-one lifts)**.** *Let $\bar{x} = P(x)$ be semisimple over an algebraically closed field, $\mu^2 \neq 1$, $\mu \neq 0$. Every $A \in \mathrm{GL}(E_\mu)$ is $P(\gamma)|_{E_\mu}$ for some $\gamma \in \mathrm{GSpin}(V)$ commuting with x and acting trivially on every E_ν, $\nu \neq \mu, \mu^{-1}$.*

Proof. For isotropic u, w and $1 + s\,\mathrm{polar}(u,w) \neq 0$, the element $\ell = 1 + s\,uw \in C(V)$ lies in $\mathrm{GSpin}(V)$ with $P(\ell)v = v + s\,\mathrm{polar}(w,v)\,u - t\,\mathrm{polar}(u,v)\,w$, $t = s/(1 + s\,\mathrm{polar}(u,w))$. For $u \in E_\mu$, $w \in E_{\mu^{-1}}$ it commutes with x ($xu = \mu ux$, $xw = \mu^{-1}wx$), and on E_μ it is $v \mapsto v + s\,\mathrm{polar}(w,v)\,u$. With a basis of E_μ and the dual basis of $E_{\mu^{-1}}$, these give all transvections and all elementary diagonal matrices, which generate $\mathrm{GL}(E_\mu)$. □

Theorem 11 (Regular nilpotent orbits and quasi-splitness)**.** $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{gspin}(V)) \neq \emptyset$ *iff* $\mathrm{GSpin}(V)$ *is quasi-split, i.e.* $\dim V - 2w(V) \leq 2$.

Proof. $\Leftarrow$: a split family $w_{-k}, \dots, w_k$ with $\dim V \leq 2k+2$ gives the explicit regular nilpotent $\sum_i \iota(w_{i+1})\iota(w_{-i})/2\nu$, whose centralizer has dimension $\lfloor \dim V/2 \rfloor + 1$, the rank (Lemma 35). $\Rightarrow$: for a regular nilpotent X, $N = \mathrm{ad}_V X$ has $\dim \mathfrak{so}(V)_N = \lfloor n/2 \rfloor$ (the rank of $\mathfrak{gspin}(V)$ is $\lfloor n/2 \rfloor + 1$), so over $\bar{F}$ it has minimal centralizer dimension among nilpotents (Lemma 35), hence partition $[n]$ or $[n-1,1]$ (Lemma 36); hence $N^m \neq 0$ with $m = n-1$ or $n-2$, and the upper half of a Jordan chain of N is a totally isotropic subspace of dimension $\lfloor (n-1)/2 \rfloor$ (Thesis §5.1.4). □

1.2 GGP triples

Definition 12 (Admissible pairs)**.** A GGP datum is a quadratic space W on F^m together with $r \geq 0$ and $\nu \in F^\times$. Put Z the split space of dimension $2r+1$ with basis $z_{-r}, \dots, z_r$, $b(z_i, z_j) = \nu\delta_{i,-j}$, and $V = W \oplus^\perp Z$ (Thesis §5.2).

Definition 13 (The group G)**.** $G = \mathrm{GSpin}(W) \times \mathrm{GSpin}(V)$ (Thesis §5.2), as a subgroup of $(C(W) \times C(V))^\times$. Its Lie algebra is $\mathfrak{g} = \mathfrak{gspin}(W) \oplus \mathfrak{gspin}(V)$, and A_G denotes its split centre.

Definition 14 (The unipotent subgroup N)**.** $N \subset \mathrm{GSpin}(V)$ is the unipotent radical of the stabilizer P_V in $\mathrm{GSpin}(V)$ of the isotropic flag $\langle z_r \rangle \subset \langle z_r, z_{r-1} \rangle \subset \cdots \subset \langle z_r, \dots, z_1 \rangle$ (Thesis §5.2).

Definition 15 (The subgroup H)**.** $H = \mathrm{GSpin}(W) \ltimes N \subset G$: the pairs (a, an) with $a \in \mathrm{GSpin}(W)$, $n \in N$ (Thesis §5.2). $Z_{G,H} = A_{G,H}$ is the diagonal central $\mathbb{G}_m$.

Definition 16 (The character ξ)**.** $\xi(h) = \psi(\lambda(h))$, where $\lambda(an) = \sum_{i=0}^{r-1} b(z_{-i-1}, nz_i)$ is a homomorphism $H \to \mathbb{G}_a$ (Thesis §5.2).

Definition 17 (Quasi-split triples)**.** G is quasi-split iff both W and V are, i.e. $\dim U - 2w(U) \leq 2$ for $U = W, V$ (Thesis §6.4).

Definition 18 (Central characters)**.** A continuous unitary character $\chi : A_G(F) \to \mathbb{S}^1$ trivial on $A_{G,H}(F)$ (Thesis §7.1). It is fixed throughout the trace formula.

Definition 19 (Whittaker data)**.** A Whittaker datum of G (Thesis §4.4): a Borel subgroup B (stabilizer of full maximal isotropic flags of W and V) with unipotent radical U, and a generic continuous unitary character χ of $U(F)$, i.e. its stabilizer in $B(F)$ is $Z_G(F)U(F)$. (The Thesis takes $G(F)$-conjugacy classes of such pairs.)

Theorem 20 (Luo's orbit $\mathcal{O}_\nu$)**.** *Let G be quasi-split. If $\dim V$ is even and ≥ 4, some regular nilpotent $X \in \mathfrak{gspin}(V)$ has image in $\mathfrak{so}(V)$ in the orbit $\mathcal{O}_\nu$ (it kills an anisotropic line Fd_0 and is one Jordan block on $U_1 = d_0^\perp \cong \langle \nu \rangle \oplus H^k$). If $\dim W$ is even and ≥ 4, some regular nilpotent $X \in \mathfrak{gspin}(W)$ has image in $\mathcal{O}_{-\nu}$.*

Proof. By Witt theory the relevant class is ν: if $\dim V$ is even, W is odd and quasi-split, hence $W = \langle a \rangle \oplus W_1$ with W_1 hyperbolic and $V = \langle a \rangle \oplus (W_1 \oplus Z)$, $W_1 \oplus Z \cong \langle \nu \rangle \oplus H^k$; if $\dim W = 2m' \geq 4$, Witt cancellation gives $W \cong \langle b \rangle \oplus \langle -\nu \rangle \oplus H^{m'-1}$. On the resulting split family the explicit element $\sum_i \iota(w_{i+1})\iota(w_{-i})/2\nu$ is regular nilpotent (its centralizer has dimension the rank, Lemma 35) and lies in $\mathcal{O}_\nu$, resp. $\mathcal{O}_{-\nu}$, by a direct computation (these are the classes used in [17], §7.1; Thesis §5.1.4). □

Theorem 21 (Germs vanish off quasi-split centralizers)**.** *For semisimple $x \in \mathrm{GSpin}(W)(F)$ embedded diagonally in G, with $W'_x = \ker(P(x)-1)$ and $V'_x = W'_x \oplus Z$: if $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x) \neq \emptyset$ then W'_x and V'_x are quasi-split.*

Proof. $\mathfrak{g}_x = \mathfrak{gspin}(W)_x \times \mathfrak{gspin}(V)_x$, and a regular nilpotent X minimizes the centralizer dimension in each factor separately. In a factor with fixed space A, $\dim \mathfrak{gspin}_{y,Y} = \dim \mathfrak{so}(A)_{\mathrm{ad}_V Y|_A} + c + 1$ where c depends only on $\mathrm{ad}_V Y$ on $A^\perp$, and the A-component of Y can be replaced by any skew map of A. So $\mathrm{ad}_V X|_A$ is a nilpotent of minimal centralizer dimension $\lfloor \dim A/2 \rfloor$ in $\mathfrak{so}(A)$, and A is quasi-split as in Theorem 11 (Lemmas 35 and 36). □

Theorem 22 (Regular nilpotent orbits of GGP centralizers)**.** *For semisimple $x \in \mathrm{GSpin}(W)(F)$ whose moved part G''_x is a torus (true for every x contributing to $\Gamma(G,H)$): $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x) \neq \emptyset$ iff W'_x and $V'_x = W'_x \oplus Z$ are quasi-split.*

Proof. $\Rightarrow$ is Theorem 21. $\Leftarrow$: the pair of explicit regular nilpotents of $\mathfrak{gspin}(W'_x)$ and $\mathfrak{gspin}(V'_x)$ is regular nilpotent in $\mathfrak{g}_x$, because with an abelian moved part the moved contribution to $\dim \mathfrak{g}_{x,Y}$ is the same for every Y. □

Literature 23 (Multiplicity at most one)**.** For every GGP triple and every irreducible tempered unitary representation π of $G(F)$, $m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi^\infty, \xi) \leq 1$ (Thesis Thm. 5.5.1).

Assumed from: [7], Thm. 1.3 ($r = 0$), with the reduction of [5], §6.3 and [9], §15; directly [28].

Chapter 2

Representations, characters and quasi-characters

Background of Thesis Ch. 2–4: tempered representations, the multiplicity $m(\pi)$, Harish-Chandra characters, quasi-characters and their germs.

2.1 Representations and the multiplicity

Definition 24 (Unitary representations)**.** A strongly continuous unitary representation of a topological group on a complex Hilbert space (Thesis §3.1).

Definition 25 (Tempered representations)**.** π is irreducible and tempered: irreducible and weakly contained in the left regular representation of G (Thesis Ch. 1, §3.1.2). The Thesis uses the growth condition on matrix coefficients and notes the equivalence (Cowling–Haagerup–Howe); the weak-containment form is used here because it stays valid when the centre of G is noncompact.

Definition 26 (The multiplicity $m(\pi)$)**.** For a unitary representation π of $G(F)$,

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi^\infty, \xi) \in \mathbb{N} \cup \{\infty\},$$

where π^∞ is the space of smooth vectors (Thesis §5.5).

Literature 27 (Schur's lemma)**.** For a continuous unitary representation of a topological group with no closed invariant subspaces other than 0 and the whole space, every bounded operator commuting with the group is a scalar.

Assumed from: [6], 13.1.4 (i)⇒(ii).

Theorem 28 (The centre acts by a character)**.** *In an irreducible unitary representation of a topological group, the centre acts by a character (Thesis §3.1.2).*

Proof. A central element's operator commutes with the whole group, so it is a scalar by Lemma 27. □

Theorem 29 ($G(F)$ is unimodular)**.** *The Haar measure of* $G(F) = \mathrm{GSpin}(W)(F) \times \mathrm{GSpin}(V)(F)$ *(and of every inner form) is right invariant.*

Proof. The modular character Δ is a homomorphism to $\mathbb{R}_{>0}$, trivial on the centre and invariant under continuous involutive automorphisms. Conjugation by an anisotropic vector w on one factor is a continuous involutive automorphism sending uw to $q(u)q(w)(uw)^{-1}$, so $\Delta(uw)^2 = 1$ and $\Delta(uw) = 1$. Scalars and products uw of anisotropic vectors generate $\mathrm{GSpin}(U)(F)$ (Cartan–Dieudonné), so $\Delta = 1$. (The thesis takes unimodularity as standard, §2.1.) □

Literature 30 (Weil's quotient-measure formula)**.** Let $S \hookrightarrow \Gamma$ be a closed embedding of unimodular locally compact, Hausdorff, second countable groups with Haar measures μ, ν. There is a measure m on $S\backslash\Gamma$ with $\int_\Gamma \varphi\, d\nu = \int_{S\backslash\Gamma}\int_S \varphi(sq)\, d\mu(s)\, dm(q)$ for all $\varphi \in C_c(\Gamma)$, unique among the measures finite on compact sets (the Radon measures). (Used for $H(F)/A_{G,H}(F) \subset G(F)/A_G(F)$, Thesis §7.1.)

Assumed from: Bourbaki, *Intégration* VII §2; [8], Thm. 2.49.

Theorem 31 (Uniqueness of the Weil measure)**.** *The measure m of Literature 30 is unique among all Borel measures.*

Proof. Any measure satisfying the formula is finite on compact sets: lift a compact set to Γ and test the formula on a bump function that is 1 on a neighbourhood of the lift. □

Literature 32 (Integration over $G = PK$)**.** Let G be a locally compact, Hausdorff, second countable group, P a closed subgroup and K a compact subgroup with $G = PK$. For Haar measures dx, dp, dk there is $c > 0$ with $\int_G f(x)\, dx = c\int_K\int_P f(pk)\, dp\, dk$ for continuous compactly supported f.

Assumed from: [13], Thm. 8.32; Bourbaki, *Intégration* VII §2; Casselman's notes §3.1.

2.2 Characters and quasi-characters

Definition 33 (The Weyl discriminant)**.** $D^G(x) = |\det(1 - \mathrm{Ad}x)|_{\mathfrak{g}/\mathfrak{g}_x}|_F$ (Thesis §2.1), computed as the absolute value of the trailing coefficient of the characteristic polynomial of $1 - \mathrm{Ad}x$ on $\mathfrak{g}$.

Literature 34 (Harish-Chandra's regular set)**.** For $x \in G(F)$ write $\det(t - 1 + \mathrm{Ad}(x)) = D(x)t^\ell + \ldots$ (terms of higher degree), $\ell = \operatorname{rank} G$. Then $G' = \{x : D(x) \neq 0\}$ is open and dense in $G(F)$ and its complement has Haar measure zero.

Assumed from: [10], Part V §3, p. 54 (also Introduction, p. 2), for G connected reductive; here $G = \mathrm{GSpin}(W) \times \mathrm{GSpin}(V)$, $\ell = (\lfloor m/2 \rfloor + 1) + (\lfloor (m + 2r + 1)/2 \rfloor + 1)$.

Literature 35 (Centralizers in $\mathfrak{so}(V)$ have dimension at least the rank)**.** Over an algebraically closed field of characteristic 0, for $n \geq 3$ and $X \in \mathfrak{so}_n$, $\dim \mathfrak{so}_n(X) \geq \lfloor n/2 \rfloor$.

Assumed from: [12], §7.11, Remark, and §7.13.

Literature 36 (Regular nilpotents of $\mathfrak{so}(V)$: Jordan type)**.** Over an algebraically closed field of characteristic 0, for $n \geq 3$, a nilpotent $X \in \mathfrak{so}_n$ whose centralizer has minimal dimension among nilpotent elements has partition $[n]$ (n odd) or $[n-1, 1]$ (n even): $X^{n-1} \neq 0$, resp. $X^{n-2} \neq 0 = X^{n-1}$.

Assumed from: [12], §6.3 (regular $\Leftrightarrow$ minimal centralizer dimension; partitions of the regular orbit).

Lemma 37 (Centralizers of regular elements)**.** *For $x \in G_{\mathrm{reg}}(F)$, $\dim \mathfrak{g}_x = \operatorname{rank} \mathfrak{g} = \min_{X \in \mathfrak{g}} \dim \mathfrak{g}_X$.*

Proof. Both sides split over $\mathfrak{g} = \mathfrak{gspin}(W) \times \mathfrak{gspin}(V)$. For $g \in \mathrm{GSpin}(U)$, $\mathfrak{gspin}(U)_g \cong F \oplus \mathfrak{so}(U)_{P(g)}$; if g is semisimple with $\mathfrak{gspin}(U)_g$ abelian then, over $\bar{F}$, the eigenvalues $\mu \neq \pm 1$ of $P(g)$ are simple and pair off, $\dim E_{\pm 1} \leq 2$, $\dim E_{-1}$ is even, and $\dim \mathfrak{so}(U)_{P(g)} = \lfloor \dim U/2 \rfloor$ by induction over orthogonal splittings; the dimension descends to F. Finally $\operatorname{rank} \mathfrak{gspin}(U) = \lfloor \dim U/2 \rfloor + 1$: Jantzen's lower bound [12], and an explicit skew element $\sum c_i J_i$ on orthogonal planes. □

Lemma 38 ($G_{\mathrm{reg}}(F) \subseteq G'$)**.** *For $x \in G_{\mathrm{reg}}(F)$, $D(x) \neq 0$.*

Proof. $N = 1 - \mathrm{Ad}x$ is semisimple on $\mathfrak{g}$, so $\mathfrak{g} = \ker N \oplus \operatorname{im} N$ with $\ker N = \mathfrak{g}_x$; the characteristic polynomial of N is $t^{\dim \mathfrak{g}_x}$ times a polynomial with nonzero constant term $\pm \det(N|_{\operatorname{im} N})$, and $\dim \mathfrak{g}_x = \ell$. □

Lemma 39 ($G' \subseteq G_{\mathrm{reg}}(F)$)**.** *If $D(x) \neq 0$ then x is semisimple with abelian centralizer in $\mathfrak{g}$.*

Proof. $D(x) \neq 0$ says the generalized 1-eigenspace of $\mathrm{Ad}x$ on $\mathfrak{g}$ has dimension $\leq \ell$. For the Jordan decomposition $x = su$ in $G(F)$, $\mathfrak{g}_s$ lies in it, so $\dim \mathfrak{g}_s \leq \ell$; the rank clauses for GSpin give $\dim \mathfrak{g}_s = \ell$, $\mathfrak{g}_s$ abelian, and the only unipotent element of $G(F)$ commuting with s is 1, so $u = 1$ and $x = s$ is regular semisimple. □

Theorem 40 ($G' = G_{\mathrm{reg}}(F)$)**.** *In characteristic 0, $D(x) \neq 0$ if and only if x is semisimple with abelian centralizer in $\mathfrak{g}$.*

Proof. The two inclusions above. □

Theorem 41 (The regular set)**.** *The regular semisimple set $G_{\mathrm{reg}}(F)$ is open in $G(F)$ and its complement has Haar measure zero.*

Proof. Harish-Chandra's theorem and the identification $G' = G_{\mathrm{reg}}(F)$. □

Literature 42 (Harish-Chandra's regularity theorem)**.** For an irreducible tempered π, the distribution $f \mapsto \operatorname{tr} \pi(f)$ on $C_c^\infty(G(F))$ is given by a locally integrable function which is locally constant on $G_{\mathrm{reg}}(F)$.

Assumed from: Bernstein (admissibility); [11]; [5], §2.1; Thesis Ch. 1.

Definition 43 (The character θ_π)**.** The Harish-Chandra character θ_π of π, the representative of $f \mapsto \operatorname{tr} \pi(f)$ that is locally constant on $G_{\mathrm{reg}}(F)$ (Thesis Ch. 1).

Definition 44 (Fourier transforms of nilpotent orbits)**.** For a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}(F)$, $\hat{j}(\mathcal{O}, \cdot)$ is the function representing $f \mapsto \mu_{\mathcal{O}}(\hat{f})$, the Fourier transform being normalized by b and ψ ([5] §1.8, §4.1; Thesis §4.1.1). It is unique on $\mathfrak{g}_{\mathrm{reg}}$.

Definition 45 (Quasi-characters)**.** $\theta : G(F) \to \mathbb{C}$ is a quasi-character (Thesis §4.1.1, [5] §4.1) if it is conjugation invariant on $G_{\mathrm{reg}}(F)$ and at every semisimple x there are coefficients $c_{\theta,\mathcal{O}}(x)$, $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)$, with $\theta(xe^X) = \sum_{\mathcal{O}} c_{\theta,\mathcal{O}}(x)\, \hat{j}(\mathcal{O}, X)$ for $X \in \mathfrak{g}_x(F)$ regular and near 0.

Definition 46 (Germs)**.** For a quasi-character θ, semisimple x and $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, the germ $c_{\theta,\mathcal{O}}(x)$ is the coefficient of $\hat{j}(\mathcal{O}, \cdot)$ in the local expansion of θ at x (Thesis §4.2, [5] §4.5); it is 0 if $\mathcal{O}$ is not regular or there is no expansion.

Literature 47 (Uniqueness of germs)**.** At semisimple x, the coefficients of a local expansion $\theta(xe^X) = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{j}(\mathcal{O}, X)$ are unique on the regular orbits $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ (linear independence of the $\hat{j}(\mathcal{O}, \cdot)$ near 0).

Assumed from: [11]; [5], §4.5; Thesis §4.2.

Literature 48 (Local character expansion)**.** For irreducible tempered π, θ_π is a quasi-character.

Assumed from: [11]; [26], §4; [5], Prop. 4.1.1(i); Thesis Prop. 4.1.1.1(i).

Literature 49 (Rodier's theorem)**.** Let G be quasi-split. For each $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$ there is a Whittaker datum (U, χ) such that for every irreducible tempered π, $c_{\theta_\pi, \mathcal{O}}(1) = 1$ if π has a (U, χ)-Whittaker model and 0 otherwise (Thesis Prop. 4.4.1(i)).

Assumed from: [19]; [18]; [5], Prop. 4.8.1(i).

Literature 50 (Self-dual Haar measures)**.** For a finite-dimensional F-vector space V with a non-degenerate bilinear form B there is a Haar measure μ on V for which Fourier inversion holds: $f(Z) = \int \widehat{f}(Y)\psi(-B(Z,Y))\, d\mu(Y)$ with $\widehat{f}(Y) = \int f(X)\psi(B(X,Y))\, d\mu(X)$, for all $f \in C_c^\infty(V)$.

Assumed from: Tate's thesis; [27], Ch. II §5, Ch. VII §2; [5], §1.6; Thesis §2.4.

Chapter 3

Inner forms and the main theorem

Thesis §6.1–6.4. The Langlands parameter is not modelled: the main theorem is proved for every family of packets satisfying the stability, transfer and genericity hypotheses, which is all the proof uses.

3.1 Inner forms, stability and packets

Definition 51 (Inner twists)**.** An inner twist of $\mathrm{GSpin}(W)$: a quadratic space W' on F^m of the same dimension and discriminant with a fixed $\overline{F}$-isometry $\varphi : W_{\overline{F}} \simeq W'_{\overline{F}}$ such that $\varphi^{-1}\circ\sigma(\varphi) \in \mathrm{SO}(W)(\overline{F})$ for all σ. It gives $\psi_\alpha = C(\varphi) \times C(\varphi \oplus \mathrm{id}_Z) : G_{\overline{F}} \simeq G_{\alpha,\overline{F}}$. This differs from Thesis §6.2: the inner forms are indexed by $H^1(F, \mathrm{SO}(W))$, not $H^1(F, \mathrm{PGSpin}(W))$.

Definition 52 (Stable conjugacy across inner forms)**.** $x \in G(F)$ and $y \in G_\alpha(F)$ are stably conjugate if $y = \psi_\alpha(gxg^{-1})$ for some $g \in G(\overline{F})$, componentwise.

Definition 53 (The Kottwitz sign)**.** $e(G_\alpha) = (-1)^{\mathrm{rk}_F G - \mathrm{rk}_F G_\alpha} = (-1)^{w(W)+w(W_\alpha)+w(V)+w(V_\alpha)}$ (the p-adic formula; that it equals Kottwitz's cohomological sign is [14]).

Definition 54 (Systems of inner forms)**.** An index set $\{\alpha\}$ with inner twists $(W_\alpha, \varphi_\alpha)$, $\alpha = 1$ the trivial one, such that every nondegenerate form on F^m with the discriminant of W is isometric to exactly one W_α; so the index set is $H^1(F, \mathrm{SO}(W)) = H^1(F, H/Z_{G,H})$. $(G_\alpha, H_\alpha, \xi_\alpha)$ is the GGP triple of $(W_\alpha, V_\alpha = W_\alpha \oplus Z)$. This differs from Thesis §6.2: the inner forms are indexed by $H^1(F, \mathrm{SO}(W))$, not $H^1(F, \mathrm{PGSpin}(W))$.

Definition 55 (Stable functions)**.** (STAB) θ is constant on stable conjugacy classes of regular semisimple elements (Thesis §6.1).

Definition 56 (Tempered packets on the inner forms)**.** Finite families Π_α of pairwise inequivalent irreducible tempered representations of $G_\alpha(F)$ with central characters trivial on $Z_{G,H}(F)$, such that, with $\theta_\alpha = \sum_{\pi\in\Pi_\alpha} \theta_\pi$ (Thesis §6.3): (STAB) each θ_α is stable; (TRANS) $\theta_\alpha(y) = e(G_\alpha)\theta_1(x)$ for stably conjugate regular $x \in G(F)$, $y \in G_\alpha(F)$; (WHITT) for every Whittaker datum of G exactly one member of Π_1 has a Whittaker model.

3.2 The main theorem

Definition 57 (The multiplicity formula (statement))**.** The statement of Thesis Thm. 7.2.2 (= Thm. 1.0.2): for every GGP triple and every irreducible tempered π with central character trivial on $A_{G,H}(F)$, $m(\pi) < \infty$ and $m(\pi) = m_{\mathrm{geom}}(\theta_\pi)$.

Theorem 58 (The multiplicity formula)**.** *The multiplicity formula holds (Thesis Thm. 7.2.2 = Thm. 1.0.2).*

Proof. By the strong induction on $\dim G$ of Thesis §7.2, from the geometric expansion, the elliptic form of Thesis Prop. 7.4.1(ii) and elliptic isolation (Chapter 5). □

Theorem 59 (The count (Thesis (6.7.1)))**.** *Let G be quasi-split and (Π_α) as above, with finitely many inner forms. Then $\sum_\alpha \sum_{\pi\in\Pi_\alpha} m(\pi) = 1$.*

Proof. By (6.7.7) the sum is the germ $c_{\theta_1,\mathcal{O}_1}(1)$. By Rodier's theorem applied to the regular orbit $\mathcal{O}_1$, $c_{\theta_1,\mathcal{O}_1}(1)$ is the number of members of Π_1 with a Whittaker model for the corresponding datum, which is 1 by (WHITT). The multiplicity formula gives $m(\pi) < \infty$. □

Theorem 60 (Main theorem: the local GGP conjecture for GSpin)**.** *(Thesis Thm. 6.4.1 = Thm. 1.0.1, p-adic case.) Let G be quasi-split and $(\Pi_\alpha)_\alpha$ tempered packets on the inner forms G_α, $\alpha \in H^1(F, \mathrm{SO}(W))$, with central characters trivial on $Z_{G,H}(F)$, satisfying (STAB), (TRANS), (WHITT). Then exactly one $\pi \in \bigsqcup_\alpha \Pi_\alpha$ has $m(\pi) = 1$, and $m(\pi') = 0$ for all others.*

Proof. The inner forms are finite in number (classification of p-adic quadratic forms). The sum of the multiplicities is 1, and each is a natural number, so exactly one is 1 and the rest vanish (Thesis §6.7). Multiplicity one is not needed for this deduction. □

Chapter 4

Proof of the main theorem from the multiplicity formula

Thesis §6.5–6.7: summing the multiplicity formula over all inner forms and rearranging the geometric side over stable classes in $\Gamma(G,H)$, everything cancels except the contribution of $x=1$. For $x \in \mathrm{GSpin}(W)(F)$ write $W'_x = \ker(P(x)-1)$, W''_x its orthogonal complement, and $T_x \subset \mathrm{GSpin}(W)$ the centralizer of x acting trivially on W'_x; $S_x = T_x/A_{G,H}$.

4.1 The space $\Gamma(G,H)$ and the geometric multiplicity

Definition 61 (Elements of $\Gamma(G,H)$)**.** $x \in \mathrm{GSpin}(W)(F)$ defines an element of $\Gamma(G,H)$ if x is semisimple, T_x is a torus and $T_x(F)/A_{G,H}(F)$ is compact. This differs from Thesis §6.5, which requires T_x itself to be anisotropic.

Definition 62 (Classes of tori in $\Gamma(G,H)$)**.** The index set of $\Gamma(G,H)$: $\mathrm{GSpin}(W)(F)$-conjugacy classes of the tori T_x, $x \in \Gamma(G,H)$.

Lemma 63 (Finiteness of the classes of tori)**.** $\Gamma(G,H)$ *meets finitely many conjugacy classes of tori (Thesis §6.5).*

Proof. Twisting argument (Serre, *Cohomologie galoisienne* I §5). Tori S_y with the same $\dim W''_y$ are conjugate under $\mathrm{SO}(W)(\bar{F})$: $S_y(\bar{F})$ is the split torus of a hyperbolic frame of eigenvectors of $P(y)$, and Witt's extension theorem carries one frame to another of the same size; a conjugator g gives the cocycle $\sigma \mapsto g^{-1}\sigma(g)$ in $N_x = N_{\mathrm{SO}(W)}(S_x)$, and $H^1(F,N_x)$ is finite. Equal classes give a Galois-fixed, hence rational, conjugator in $\mathrm{SO}(W)(F)$, which lifts to $\mathrm{GSpin}(W)(F)$ and conjugates the tori. □

Definition 64 (The function Δ)**.** $\Delta(x) = |\det((P(x)-1)|_{W''_x})|_F$ (Thesis (6.5.9)).

Definition 65 (The orbit $\mathcal{O}_x$)**.** The regular nilpotent orbit $\mathcal{O}_x \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ selected by Luo's recipe ([17] §7.1) applied to (G'_x, H'_x); a junk value if there is none (then germs vanish anyway). This differs from the choice in Thesis §5.4: it is Luo's recipe applied to (G'_x, H'_x).

Definition 66 (The geometric multiplicity m_{geom})**.** For $s > 0$,

$$m_{\mathrm{geom},s}(\theta) = \sum_{[T]} |W(T)|^{-1} \int_{T(F)/A_{G,H}(F)} D^G(t)^{1/2} c_{\theta,\mathcal{O}_t}(t) \Delta(t)^{s-1/2}\, dt,$$

the sum over the classes of tori of $\Gamma(G,H)$, with the probability Haar measure and $W(T) = N_{\mathrm{GSpin}(W)(F)}(T)/Z_{\mathrm{GSpin}(W)(F)}(T)$; and $m_{\mathrm{geom}}(\theta) = \lim_{s\to 0^+} m_{\mathrm{geom},s}(\theta)$ (Thesis §7.2, Prop. 12.1.1). This differs from Thesis (6.5.3), which uses $N_{\mathrm{GSpin}(W'')(F)}(T)/T(F)$ for $W(T)$.

Proposition 67 (Linearity of m_{geom} on characters)**.** *For tempered $\pi_1, \dots, \pi_k$ with central characters trivial on $A_{G,H}(F)$, $m_{\mathrm{geom}}(\sum_i \theta_{\pi_i}) = \sum_i m_{\mathrm{geom}}(\theta_{\pi_i})$ (Thesis Prop. 12.1.1, used in (6.7.2)).*

Proof. Each θ_{π_i} lies in $\mathrm{QC}(G,\eta_i)$ for its central character η_i, and m_{geom} is linear on the span of these spaces (Chapter 7). □

Proposition 68 (Thesis (6.7.2))**.** $\sum_{\pi\in\Pi_\alpha} m(\pi) = m_{\mathrm{geom}}(\theta_\alpha)$ *for every inner form α.*

Proof. The multiplicity formula for $(G_\alpha, H_\alpha, \xi_\alpha)$ and linearity of m_{geom}. □

4.2 Germ transfer

Lemma 69 (Finitely many regular nilpotent orbits)**.** *For semisimple $x \in G(F)$ the set $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ of regular nilpotent $G_x(F)$-orbits is finite.*

Proof. Not yet formalized. The regular nilpotent elements form one geometric orbit, whose rational orbits are indexed by a subset of $H^1(F, Z_{G_x}(N))$, finite by Borel–Serre (1964), Thm. 6.1. □

Lemma 70 (Regular germs of a stable quasi-character)**.** *If θ is a stable quasi-character on $G(F)$ and $x \in G(F)$ is semisimple, then $c_{\theta,\mathcal{O}}(x) = c_{\theta,\mathcal{O}'}(x)$ for all $\mathcal{O},\mathcal{O}' \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.*

Proof. Not yet formalized. As in the proof of [5], (3.4.7): $G_{x,\mathrm{ad}}(F)$ acts transitively on $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, stability makes $X \mapsto \theta(xe^X)$ invariant under it, and the local expansion is unique. □

Theorem 71 (Transfer preserves average germs)**.** *(Thesis (6.1.2).) Let G be quasi-split, θ a stable quasi-character on $G(F)$ and θ' a stable quasi-character on $G_\alpha(F)$ with $\theta'(y) = \theta(x)$ for stably conjugate regular x, y. Then for semisimple, strongly stably conjugate $x \in \mathrm{GSpin}(W)(F)$, $y \in \mathrm{GSpin}(W_\alpha)(F)$, $c_{\theta'}(y) = c_\theta(x)$, where $c_\theta(x)$ is the average of the regular germs.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [5], (12.1.2), stated there for pure inner forms; its proof does not use purity. □

Theorem 72 (Transfer preserves germs)**.** *(Thesis (6.1.2).) Under the hypotheses of Theorem 71, $c_{\theta',\mathcal{O}'}(y) = c_{\theta,\mathcal{O}}(x)$ for all regular $\mathcal{O},\mathcal{O}'$.*

Proof. Each regular germ of θ and of θ' equals the average (Lemmas 70 and 69); apply Theorem 71. □

Proposition 73 (Germ transfer for packets)**.** *If $x \in \mathrm{GSpin}(W)(F)$, $y \in \mathrm{GSpin}(W_\alpha)(F)$ are semisimple and strongly stably conjugate, then $c_{\theta_\alpha,\mathcal{O}'}(y) = e(G_\alpha)\, c_{\theta_1,\mathcal{O}}(x)$ for all regular $\mathcal{O},\mathcal{O}'$ (Thesis §6.7 after (6.7.4)).*

Proof. Apply the germ transfer to $e(G_\alpha)\theta_1$ and θ_α, using (STAB) and (TRANS). □

4.3 Stable conjugacy in $\Gamma(G, H)$

Thesis Prop. 6.6.1.

Lemma 74 (Thesis Prop. 6.6.1(i), step 1)**.** *Let G be quasi-split and $y \in \Gamma(G_\alpha, H_\alpha)$ with $G_{\alpha,y}$ quasi-split. Then $W'_{\alpha,y}$ embeds isometrically in W.*

Proof. By the classification of p-adic quadratic forms and Witt cancellation: if $W''_{\alpha,y}$ is 0 or a hyperbolic plane then $W_\alpha \cong W$; $\dim W''_{\alpha,y} = 1$ is impossible; otherwise a second class of the dimension and discriminant of $W''_{\alpha,y}$ exists. □

Lemma 75 (Thesis Prop. 6.6.1(i), steps 2–3)**.** *Given an isometric embedding $W'_{\alpha,y} \hookrightarrow W$, some $x \in \Gamma(G, H)$ is strongly stably conjugate to y.*

Proof. W is the twist of W_α by a cocycle in S_y: twist $W''_{\alpha,y}$ into $\iota(W'_{\alpha,y})^\perp$ in block coordinates, a cocycle in $SO(W_\alpha)_{W'_y}$, moved into S_y by [15] Lemma 10.2. Then the construction of Prop. 6.6.1(iii) for G_α and the inverse twist (W, φ_α^{-1}), and symmetry of strong stable conjugacy. □

Proposition 76 (Thesis Prop. 6.6.1(i))**.** *Let G be quasi-split and $y \in \Gamma(G_\alpha, H_\alpha)$ with $G_{\alpha,y}$ quasi-split. Then some $x \in \Gamma(G, H)$ is strongly stably conjugate to y.*

Proof. Combine the two preceding lemmas. □

Definition 77 (Torus transfer)**.** For $x \in \Gamma(G, H)$, $y \in \Gamma(G_\alpha, H_\alpha)$: an F-rational isomorphism of topological groups $T_x(F) \simeq T_y(F)$, the identity on $A_{G,H}$, with $x \mapsto y$, matching the regular parts $T_\natural$ and the strong stable classes, and preserving D^G and Δ (the conclusion of Thesis Prop. 6.6.1(ii)). It need not preserve $|W(T)|$.

Lemma 78 (The torus isomorphism)**.** *If $x \in \Gamma(G, H)$ and $y \in \Gamma(G_\alpha, H_\alpha)$ are strongly stably conjugate, then $\mathrm{Ad}(g) \circ \psi_\alpha$ restricts to an F-rational isomorphism of topological groups $T_x(F) \simeq T_y(F)$, the identity on scalars, $x \mapsto y$, carrying $T_{x,\natural}$ onto $T_{y,\natural}$ with every t stably conjugate to its image.*

Proof. The values of $t \mapsto gC(\varphi_\alpha)(t)g^{-1}$ are Galois-fixed, hence rational (Galois descent); they commute with y and fix W'_y. The inverse comes from g^{-1} and φ_α^{-1}. Scalars are central and $x \mapsto y$ by the choice of g. e and e^{-1} are restrictions of F-linear maps $C(W) \to C(W_\alpha)$, hence continuous. Stable conjugacy in G is Lemma 79. On $T_{x,\natural}$ the centralizers of $\bar{t}$ and $\bar{x}$ in $\mathrm{GSpin}(W)(\overline{F})$ and $\mathrm{GSpin}(V)(\overline{F})$ agree (by the centralizer theorem, Theorem 7, and the commutativity of the Lie algebra of S_t); hence the rationality clauses for x give those for t, and $\mathrm{Ad}(g) \circ C(\varphi_\alpha)$ carries centralizers, fixed spaces and semisimplicity, so $T_{x,\natural}$ and $T_{y,\natural}$ correspond. □

Lemma 79 (Stable conjugacy in G from W)**.** *If $\bar{s} = \mathrm{Ad}(g)C(\varphi_\alpha)(\bar{t})$ with $g \in \mathrm{GSpin}(W_\alpha)(\overline{F})$, then $\mathrm{diag}(t)$ and $\mathrm{diag}(s)$ are stably conjugate in G through ψ_α.*

Proof. On V conjugate by the image of g under $W_{\alpha,\overline{F}} \subseteq V_{\alpha,\overline{F}}$: the two compositions $C(W) \to C(V_{\alpha,\overline{F}})$ agree on W. □

Lemma 80 (Δ is a stable invariant)**.** *If $P(t') = hP(t)h^{-1}$ over $\overline{F}$ with $h = P(g) \circ \varphi$, then $\Delta(t') = \Delta(t)$.*

Proof. h conjugates $(P(t)-1)|_{W_t''}$ to $(P(t')-1)|_{W_{t'}''}$; both determinants lie in F. □

Lemma 81 (D^G is a stable invariant)**.** *If $t \in G(F)$ and $t' \in G_\alpha(F)$ are stably conjugate then $D^{G_\alpha}(t') = D^G(t)$.*

Proof. $\psi_\alpha \circ \mathrm{Ad}(g)$ conjugates $1 - \mathrm{Ad}t$ to $1 - \mathrm{Ad}t'$ over $\overline{F}$; characteristic polynomials are unchanged by base change. □

Proposition 82 (Thesis Prop. 6.6.1(ii))**.** *If $x \in \Gamma(G,H)$ and $y \in \Gamma(G_\alpha, H_\alpha)$ are strongly stably conjugate, there is a torus transfer from x to y. So the Γ-integrals over $T_x(F)/A(F)$ and $T_y(F)/A(F)$ match locally.*

Proof. The torus isomorphism, with the invariance of Δ and D^G. □

4.4 The signed count

Thesis (6.7.6) and Prop. 6.6.1(iii).

Definition 83 (The counts $n_\alpha(x)$)**.** $n_\alpha(x)$ is the number of $\mathrm{GSpin}(W_\alpha)(F)$-conjugacy classes of $y \in \Gamma(G_\alpha, H_\alpha)$ strongly stably conjugate to x (Thesis §6.7).

Definition 84 (The signed count)**.** $\sum_\alpha e(G_\alpha)\, n_\alpha(x)$ (Thesis (6.7.6)).

Theorem 85 (Hilbert's Theorem 90 for GL_m)**.** *Every continuous cocycle $\sigma \mapsto a_\sigma \in \mathrm{GL}_m(\overline{F})$ of $\mathrm{Gal}(\overline{F}/F)$ is a coboundary: $a_\sigma = M^{-1}\sigma(M)$.*

Proof. Speiser's argument: average the cocycle over $\mathrm{Gal}(L/F)$, L the normal closure of a field of definition, and use Dedekind's independence of characters to find an invertible average B with $a_\sigma\, \sigma(B) = B$ ([21], Ch. X §1 Prop. 3). □

Lemma 86 (The map $q_x : H^1(F, S_x) \to \{\alpha\}$)**.** *Every class $\beta \in H^1(F, S_x)$ determines a unique inner form $\alpha = q_x(\beta)$: the twist of W by a cocycle representing β is isometric to exactly one W_α.*

Proof. Hilbert 90 writes the cocycle as $M^{-1}\sigma(M)$; $v \mapsto Q(M^{-1}v)$ is an F-form of the discriminant of W. Cohomologous cocycles give F-isometric forms by Galois descent. □

Lemma 87 ($P(x)$ is semisimple over $\overline{F}$)**.** *For $x \in \Gamma(G,H)$, $P(x) \otimes \overline{F}$ is semisimple on $W_{\overline{F}}$.*

Proof. Conjugation by x is a commuting product of left and right multiplication by elements of $F[x]$, hence semisimple; it restricts to $P(x)$ on $\iota(W)$. Squarefree polynomials stay squarefree over $\overline{F}$ in characteristic 0. □

Lemma 88 (The Lie algebra of S_x is abelian)**.** *For $x \in \Gamma(G,H)$, the skew endomorphisms of $W_{\overline{F}}$ commuting with $P(x)$ and vanishing on $W_x' \otimes \overline{F}$ commute.*

Proof. A skew Y lifts to $\hat{Y} \in \mathfrak{gspin}(W_{\overline{F}})$ with $[\hat{Y}, v] = Yv$; $\hat{Y}$ commutes with x (the difference $x\hat{Y}x^{-1} - \hat{Y}$ is a traceless central element) and with $W_x' \otimes \overline{F}$. These are F-linear conditions, so by flatness the $\overline{F}$-solutions are spanned by the F-solutions, which commute by the commutativity of the moved part of $\mathfrak{g}_x$ over F. Then $[[\hat{Y}_1, \hat{Y}_2], v] = [Y_1, Y_2]v$. □

Lemma 89 (S_x is commutative)**.** *For $x \in \Gamma(G,H)$, the group $S_x(\overline{F})$ is commutative.*

Proof. $S_x(\overline{F})$ consists of the isometries of determinant 1 commuting with the semisimple $P(x)$ and fixing W'_x. If the corresponding Lie algebra is abelian, the eigenvalues $\lambda \neq \pm 1$ of $P(x)$ are simple and $\dim E_{-1} \leq 2$; the determinant on E_{-1} is 1, and Cayley–Hamilton on E_{-1} gives commutativity. □

Literature 90 (Finiteness of $H^1(F,S_x)$)**.** For $x \in \Gamma(G,H)$, $H^1(F,S_x)$ is finite.
Assumed from: [20], III §4.3 Thm. 4 (Borel–Serre); Tate–Nakayama.

Literature 91 (Fundamental tori)**.** For $x \in \Gamma(G,H)$, $H^1(F,S_x) \to H^1(F,SO(W)_{W'_x})$ is surjective (S_x is an anisotropic, hence fundamental, maximal torus of $SO(W)_{W'_x} \cong SO(W''_x)$).
Assumed from: [15], §10, Lemma 10.2.

Literature 92 (Finiteness of $H^1(F,N_x)$)**.** For $x \in \Gamma(G,H)$, $H^1(F,N_x)$ is finite, $N_x = N_{\mathrm{SO}(W)}(S_x)$.
Assumed from: Borel–Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. 39 (1964), Thm. 6.1.

Lemma 93 (Thesis Prop. 6.6.1(iii))**.** *For $x \in \Gamma(G,H)$ and every α, $n_\alpha(x) = |q_x^{-1}(\alpha)|$, $q_x^{-1}(\alpha) \subset H^1(F,S_x)$. This differs from Thesis Prop. 6.6.1(iii), which uses $H^1(F,T_x)$.*

Proof. Not yet formalized. Galois-cohomological parametrization of stable classes, then Witt cancellation. □

Theorem 94 (The Kottwitz sign is multiplicative on $H^1(F,S_x)$)**.** *For G quasi-split and $x \in \Gamma(G,H)$, $\beta \mapsto e(G_{q_x(\beta)})$ is multiplicative on $H^1(F,S_x)$.*

Proof. Own work, without Brauer groups (the statement is [5], (12.1.6), with the sign of [14]). $e(G_\alpha) = 1$ exactly when $W_\alpha \cong W$. For a cocycle a in S_x choose lifts $t_\sigma \in T_x(\bar{F})$; the scalars $t_\sigma\, \sigma(t_\tau)\, t_{\sigma\tau}^{-1}$ form its obstruction. Call $a \sim b$ when they have lifts with equal obstructions; as $T_x(\bar{F})$ is commutative this is compatible with products. If $W_\alpha \cong W_\beta$, a lift $h \in \mathrm{GSpin}(W)(\bar{F})$ of $\varphi_a^{-1}\varphi_b$ (made proper by a rational reflection) conjugates a lift of a into one of b, so $a \sim b$; conversely $a \sim b$ gives $W_\alpha \cong W_\beta$ (Lemma 95). With at most two classes of the dimension and discriminant of W, the sign is multiplicative. □

Lemma 95 (Twists with equal obstructions are isometric)**.** *Let U, U_1, U_2 be nondegenerate. If the cocycles of $\varphi_i \colon U_{\bar{F}} \cong U_{i,\bar{F}}$ have continuous lifts to $\mathrm{GSpin}(U)(\bar{F})$ with equal (scalar) obstructions, then $U_1 \cong U_2$ over F.*

Proof. Own work. The quotient $v_\sigma = \varphi_1 s_\sigma t_\sigma^{-1} \varphi_1^{-1}$ of the lifts is a 1-cocycle in $\mathrm{GSpin}(U_1)(\bar{F})$, and $C(\psi)$, $\psi = \varphi_2\varphi_1^{-1}$, satisfies $\sigma(C(\psi)z) = C(\psi)(v_\sigma \sigma(z) v_\sigma^{-1})$. Hilbert 90 for the units of the central simple algebra $C(U_1)$ (even dimension) or $C^+(U_1)$ (odd dimension), deduced from Hilbert 90 for GL_N and the fact that modules of equal dimension over a simple algebra are isomorphic, gives $v_\sigma = u^{-1}\sigma(u)$; then $z \mapsto C(\psi)(u^{-1}zu)$ is Galois-equivariant and descends to $C(U_1) \cong C(U_2)$ (resp. C^+). The discriminants agree since $\det\psi$ is Galois-fixed, so $U_1 \cong U_2$ by the classification of quadratic forms over F by rank, discriminant and Clifford invariant. □

Lemma 96 (A twist into the other form: the plane case)**.** *If W''_x contains an x-stable anisotropic plane Π on which $P(x)$ has determinant 1, some $\beta \in H^1(F,S_x)$ twists W into a form $W_{q_x(\beta)} \ncong W$.*

Proof. Over $\bar{F}$ the plane is hyperbolic with hyperbolic pair f, f'; $\mathrm{SO}(\Pi)(\bar{F})$ is the split torus scaling f, f' by t^{-1}, t, lifted to GSpin by $ff' + t\,f'f$, which commutes with x and fixes W'_x, so it lies in S_x. Twisting Π into the other binary form of its discriminant gives a cocycle with these values; the twisted form is not isometric to W by Witt cancellation. □

Lemma 97 (The moved space of $x \in \Gamma(G,H)$)**.** *For $x \in \Gamma(G,H)$, W''_x is not a line, and if it is a plane it is anisotropic.*

Proof. On a line $P(x)$ is the reflection in it, of determinant -1. On a hyperbolic plane with hyperbolic pair f, f', $P(x)$ is diagonal on f, f' (else it is a reflection), so the lifts $h(s) = ff' + s\,f'f$ of the hyperbolic torus commute with x and lie in $T_x(F)$, acting by s on f: this contradicts compactness of $T_x(F)/A(F)$. □

Lemma 98 (Twisting W''_x along S_x)**.** *For $x \in \Gamma(G,H)$ with W''_x of dimension ≥ 3 or an anisotropic plane, some $\beta \in H^1(F, S_x)$ twists W into a form $W_{q_x(\beta)} \ncong W$.*

Proof. Twist the block W''_x into the other form of its dimension and discriminant; the cocycle lies in $SO(W)_{W'_x}$, so by [15] Lemma 10.2 it is cohomologous to one in S_x; Witt cancellation. □

Lemma 99 (A twist into the other form, for $x \in \Gamma(G,H)$)**.** *For $x \in \Gamma(G,H)$ not central, some $\beta \in H^1(F, S_x)$ twists W into a form $W_{q_x(\beta)} \ncong W$.*

Proof. $W''_x \neq 0$ since x is not central; Lemmas 97 and 98. □

Lemma 100 (A twist into the other form)**.** *For G quasi-split and $x \in \Gamma(G,H)$ not central, some $\beta \in H^1(F, S_x)$ twists W into a form $W_{q_x(\beta)} \ncong W$.*

Proof. Lemma 99. □

Lemma 101 (The sign takes the value -1)**.** *For G quasi-split and $x \in \Gamma(G,H)$ not central, some $\beta \in H^1(F, S_x)$ has $e(G_{q_x(\beta)}) = -1$.*

Proof. From the previous lemma: if $W_\alpha \ncong W$ then $e(G_\alpha) = -1$, since two p-adic forms of the same dimension and discriminant have Witt indices of opposite parity in odd dimension and of the same parity in even dimension. □

Theorem 102 (The signed count)**.** *Let G be quasi-split and $x \in \Gamma(G,H)$ with G_x quasi-split. Then $\sum_\alpha e(G_\alpha) n_\alpha(x) = 1$ if x is central and 0 otherwise. This replaces the Kottwitz-sign cancellation of Thesis §6.7 after (6.7.6).*

Proof. The sum equals $\sum_{\beta \in H^1(F,S_x)} e(G_{q_x(\beta)})$. If x is central this is $e(G) = 1$. Otherwise the sign is a character of the finite group $H^1(F, S_x)$ taking the value -1, so the sum vanishes. □

4.5 Rearrangement and the atom

Lemma 103 (The torus-class count)**.** *Let f be a function of the tori T_x, $x \in \Gamma(G,H)$, with $f(T_x) = f(T_{x'})$ whenever x, x' are strongly stably conjugate to a common $y \in \Gamma(G_\beta, H_\beta)$, and $f = 0$ on non-quasi-split tori. Then*

$$\sum_{[T'] \in \Gamma(G_\alpha, H_\alpha)/\sim} |W(T')|^{-1} f(T_{x(T')}) = \sum_{[T] \in \Gamma(G,H)/\sim} |W(T)|^{-1} \frac{n_\alpha(x_T)}{n_1(x_T)} f(T),$$

where $x(T')$ is strongly stably conjugate to a generator of T' ($f(T_{x(T')}) := 0$ if there is none).

Proof. Not yet formalized. Regroup by stable classes; $n_\alpha(x) = |q_x^{-1}(\alpha)|$, and the rational classes of tori over α are the orbits of $(N_x/S_x)(F)$ on $q_x^{-1}(\alpha) \subseteq H^1(F, S_x)$, with stabilizers $W(T')$ (orbit–stabilizer). □

Lemma 104 (The stable rearrangement at $s > 0$)**.** *Given germ transfer and Prop. 6.6.1(i),(ii): for $s > 0$,*

$$\sum_\alpha m_{\mathrm{geom},s}(\theta_\alpha) = \int_{\Gamma(G,H)} \frac{\sum_\alpha e(G_\alpha) n_\alpha(x)}{n_1(x)} c_{\theta_1}(x) D^G(x)^{1/2} \Delta(x)^{s-1/2}\, dx$$

(Thesis (6.7.2)–(6.7.5) before the limit).

Proof. Torus by torus: non-quasi-split tori contribute 0; a quasi-split torus T_y of G_α has $\int_{T_y/A} = e(G_\alpha) \int_{T_x/A}$ for $x \sim_{\mathrm{stab}} y$ (the torus transfer identifies the probability Haar measures, D^G and Δ; germ transfer at the regular orbits $\mathcal{O}_x$, $\mathcal{O}_y$); regroup with the torus-class count; the stable weight is constant on $T_{x,\natural}$. □

Proposition 105 (Thesis (6.7.5))**.** *Under the same hypotheses,*

$$\sum_\alpha m_{\mathrm{geom}}(\theta_\alpha) = \lim_{s \to 0^+} \int_{\Gamma(G,H)} \frac{\sum_\alpha e(G_\alpha) n_\alpha(x)}{n_1(x)} c_{\theta_1}(x) D^G(x)^{1/2} \Delta(x)^{s-1/2}\, dx.$$

Proof. The rearrangement at $s > 0$, and the finite sum over α commutes with the limit. □

Lemma 106 (Central translation of the germ)**.** *For central $c \in A_{G,H}(F)$, $c_{\theta_1,\mathcal{O}_c}(c) = c_{\theta_1,\mathcal{O}_1}(1)$.*

Proof. $\theta_1(cy) = \theta_1(y)$ on $G_{\mathrm{reg}}(F)$ since the central characters are trivial on $Z_{G,H}(F)$; e^X is regular for small regular X, so the expansion at 1 is one at c (uniqueness of germs); $\mathcal{O}_c = \mathcal{O}_1$. □

Lemma 107 (The atom)**.** *Assuming the signed count, the contribution of the class of $T = A_{G,H}$ to the stable integral is $c_{\theta_1,\mathcal{O}_1}(1)$: $|W(T)| = 1$, $T(F)/A(F)$ is a point, the weight is 1 and $D^G = \Delta = 1$.*

Proof. Direct computation, with central translation of the germ. □

Proposition 108 (Only the atom contributes)**.** *If the signed count holds on quasi-split classes, then for $s > 0$ the stable integral equals $c_{\theta_1,\mathcal{O}_1}(1)$ (Thesis §6.7 after (6.7.6)).*

Proof. Non-central and non-quasi-split classes have weight 0; the atom gives the value above. □

Lemma 109 (Luo's orbit at quasi-split points)**.** *Let $x \in \mathrm{GSpin}(W)(F)$ be semisimple with abelian moved part and W'_x, V'_x quasi-split. Then some $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ satisfies Luo's condition at x ([17] §7.1).*

Proof. Luo's split family for the descended datum (W'_x, V'_x), carried into V (or W) by the isometric embedding, paired with a regular element of the other factor. □

Lemma 110 ($\mathcal{O}_x$ is regular at quasi-split points)**.** *For $x \in \Gamma(G,H)$ with G'_x quasi-split, $\mathcal{O}_x \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.*

Proof. Lemma 109. □

Lemma 111 ($\mathcal{O}_1$ is regular)**.** *For G quasi-split, $\mathcal{O}_1 \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$.*

Proof. Luo's recipe applied to (G, H) selects a regular orbit, which exists since G is quasi-split. □

Theorem 112 (Thesis (6.7.7))**.** *Let G be quasi-split. Then $\sum_{\alpha} \sum_{\pi \in \Pi_\alpha} m(\pi) = c_{\theta_1, \mathcal{O}_1}(1)$.*

Proof. Sum (6.7.2) over α; rearrange by (6.7.5) using germ transfer and Prop. 6.6.1(i),(ii); the signed count kills every class but the atom. □

Chapter 5

The multiplicity formula

Thesis Ch. 7. The multiplicity formula is proved by strong induction on $\dim G$, together with the geometric expansion of the distribution J. Test functions are taken in C_c^∞ modulo $A_G(F)$ instead of the Harish-Chandra Schwartz space; every stated theorem of the trace formula uses this space.

5.1 Strongly cuspidal functions and the distribution J

Definition 113 (Strongly cuspidal test functions)**.** $\mathcal{C}_{\mathrm{scusp}}(A_G(F)\backslash G(F), \chi^{-1})$, C_c^∞ version: locally constant f on $G(F)$ with $f(ag) = \chi(a)^{-1} f(g)$ for $a \in A_G(F)$, compactly supported modulo $A_G(F)$, and strongly cuspidal ($\int_{U(F)} f(pu)\,du = 0$ for every proper parabolic $P = MU$ and $p \in P(F)$) (Thesis Ch. 1, §4.5, §7.1).

Definition 114 (The distribution J)**.**

$$J(f) = \int_{H(F)\backslash G(F)/A_G(F)} K(f,x)\,dx, \qquad K(f,x) = \int_{H(F)/A_{G,H}(F)} f(x^{-1}hx)\xi(h)\,dh$$

(Thesis §7.1, §10.1), for the Weil quotient measure on $H(F)\backslash G(F)/A_G(F)$.

Theorem 115 (Thesis Thm. 10.1.1(ii))**.** *For every $d > 0$ there is a continuous seminorm ν_d with $|K(f,x)| \le \nu_d(f)\,\Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{-d}$ for strongly cuspidal f.*

Proof. Not yet formalized (adapted from [5] Thm. 8.1.1(ii)). □

Proposition 116 (Thesis Prop. 8.4.1(iii))**.** *For every compact $C \subseteq G(F)$ with nonempty interior defining $\Xi^{H\backslash G}$, there is $d > 0$ with $\int_{H(F)\backslash G(F)/A_G(F)} \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{-d}\,dx < \infty$.*

Proof. Not yet formalized (adapted from [5] Prop. 6.7.1(iii)). □

Proposition 117 (Iwasawa data exist)**.** *$G(F)$ has a minimal parabolic subgroup P_0 and a compact open subgroup K with $G(F) = P_0(F)K$.*

Proof. Not yet formalized (Bruhat–Tits). □

Theorem 118 (Convergence of J)**.** *For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$, $x \mapsto K(f,x)$ is integrable on $H(F)\backslash G(F)/A_G(F)$ (Thesis Thm. 10.1.1(ii) with Prop. 8.4.1(iii); = Thm. 1.0.3).*

Proof. f is Harish-Chandra Schwartz and bi-invariant under a compact open subgroup K', so $K(f,\cdot)$ is continuous on $H(F)\backslash G(F)/A_G(F)$. The bound $|K(f,x)| \le \nu(f)\,\Xi^{H\backslash G}(x)^2\sigma(x)^{-d}$ for strongly cuspidal f and every $d > 0$, with integrability of $\Xi^{H\backslash G}(x)^2\sigma(x)^{-d}$ for d large, gives integrability. □

Lemma 119 (J is linear)**.** *$J(f+g) = J(f) + J(g)$ for $f, g \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$ (Thesis §10.1).*

Proof. Linearity of the integral, given convergence. □

Definition 120 (The quasi-character θ_f)**.** For strongly cuspidal f, θ_f is the function representing the orbital pairing $f' \mapsto \int_{G(F)/A_G(F)} \int_{G(F)} f(x^{-1}gx) f'(g)\, dg\, dx$ (Thesis §4.8, [5] §5.5); 0 if there is none. Convention: $\theta_f(ag) = \chi(a)^{-1}\theta_f(g)$.

Theorem 121 (θ_f represents the orbital pairing)**.** *Let $f_0 \in C_c^\infty(G(F))$ be strongly cuspidal. Then there is a quasi-character θ, locally integrable and locally constant on $G_{\mathrm{reg}}(F)$, with $\int_{G(F)} \theta f'\, dg = \int_{G(F)/A_G(F)} \int_{G(F)} f_0(x^{-1}gx) f'(g)\, dg\, dx$ for every $f' \in C_c^\infty(G(F))$, the outer integrand being integrable.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [5], Thm. 5.5.1(ii),(iv), Prop. 5.6.1(ii) (= Thesis Thm. 4.7.1, Prop. 4.8.1(ii)); [2]; Weyl integration formula. □

Lemma 122 (χ-transfer of θ_f)**.** *For strongly cuspidal $f_0 \in C_c^\infty(G(F))$, the orbital pairing of its χ-projection $P_\chi f_0$ is represented by a quasi-character (Thesis §4.8, §7.1).*

Proof. Glue central translates of the quasi-character of a cut-off of $P_\chi f_0$ along the fibres of the Harish-Chandra map H_G. □

Proposition 123 (θ_f of a strongly cuspidal function)**.** *For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$, θ_f represents the orbital pairing of f and is a quasi-character (Thesis §4.8, Prop. 4.8.1(ii)).*

Proof. From the χ-transfer and uniqueness on the open conull regular set. □

Definition 124 ($\mathrm{QC}(G(F), \chi^{-1})$)**.** Quasi-characters θ on $G(F)$ with $\theta(ag) = \chi(a)^{-1}\theta(g)$ for $a \in A_G(F)$, $g \in G_{\mathrm{reg}}(F)$ (Thesis §4.1.1, §7.2).

Lemma 125 ($\theta_f \in \mathrm{QC}(G, \chi^{-1})$)**.** *For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$, $\theta_f \in \mathrm{QC}(G(F), \chi^{-1})$ (Thesis Prop. 4.8.1(ii)).*

Proof. From the previous proposition and central equivariance of f. □

Lemma 126 (Characters are quasi-characters)**.** *If π is irreducible tempered with central character η on $A_G(F)$ then $\theta_\pi \in \mathrm{QC}(G(F), \eta)$ (Thesis Ch. 1, §4.4).*

Proof. Harish-Chandra's local expansion, and central translation of characters. □

5.2 The induction

Thesis §7.2 and §7.4.

Definition 127 (The geometric expansion at (G,χ))**.** $J(f) = m_{\mathrm{geom}}(\theta_f)$ for every $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$ (Thesis Thm. 7.2.1 = Thm. 12.2.1 = Thm. 1.0.5).

Definition 128 (The multiplicity formula at one triple)**.** For every irreducible tempered π of $G(F)$ trivial on $A_{G,H}(F)$: $m(\pi) < \infty$ and $m(\pi) = m_{\mathrm{geom}}(\theta_\pi)$.

Definition 129 (The induction hypothesis (HYP))**.** The geometric expansion (for every central character) and the multiplicity formula hold for every GGP triple (G', H', ξ') with $\dim G' < \dim G$ (Thesis §7.2).

Definition 130 (Virtual tempered representations)**.** $R_{\mathrm{temp}}(G,\eta)\otimes\mathbb{C}$: finite formal linear combinations of classes of irreducible tempered representations with central character η on $A_G(F)$ (Thesis §3.4, §7.4), with the linear extensions θ_Π, $m(\Pi)$, $m_{\mathrm{geom}}(\Pi)$.

Definition 131 (The defect)**.** $\Pi \mapsto m(\Pi) - m_{\mathrm{geom}}(\Pi)$, extended linearly (Thesis Prop. 7.4.1(ii)). The multiplicity formula for π is $\mathrm{defect}[\pi] = 0$.

Theorem 132 (Elliptic isolation)**.** *There is an open, closed, invariant, $A_G(F)$-stable $\Omega \subset G(F)$ containing the elliptic regular set and compact modulo conjugation and $A_G(F)$, such that every quasi-character supported (compactly modulo conjugation) in Ω is θ_{f_0} on $G_{\mathrm{reg}}(F)$ for some strongly cuspidal $f_0 \in C_c^\infty(G(F))$ supported in Ω.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [5], Cor. 5.7.2(i) (= Thesis Cor. 4.8.3(i)), stated there for compact centre; adapted to a nontrivial split centre A_G by its local proof. □

Proposition 133 (Elliptic isolation, χ-equivariant)**.** *For every virtual tempered Π with central character η there is $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \eta)$ with $\theta_f = \theta_\Pi$ on $G_{\mathrm{ell,reg}}(F)$ (Thesis Cor. 4.8.3(i) in χ-equivariant form). This differs from the Thesis, which uses [5] Cor. 5.7.2(iv) and Schwartz functions.*

Proof. Cut θ_Π off near the elliptic set, apply elliptic isolation, and project with P_χ. □

Proposition 134 (Thesis Prop. 7.4.1(ii), elliptic form)**.** *Assume (HYP). Let $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \eta)$ and Π virtual tempered with central character η such that $\theta_\Pi = \theta_f$ on $G_{\mathrm{ell,reg}}(F)$. Then $J(f) - m_{\mathrm{geom}}(\theta_f) = m(\Pi) - m_{\mathrm{geom}}(\Pi)$.*

Proof. The spectral expansion gives a Π_f with the same elliptic character and $J(f) - m_{\mathrm{geom}}(\theta_f) = \mathrm{defect}(\Pi_f)$; $\Pi - \Pi_f$ has vanishing elliptic character, so it lies in R_{ind}, where the defect vanishes (Chapter 6). □

Proposition 135 (Geometric expansion implies multiplicity formula)**.** *(Thesis Prop. 7.4.1(iii), direction Thm. 7.2.1 ⇒ Thm. 7.2.2.) Under (HYP), if the geometric expansion holds at G for all central characters, the multiplicity formula holds at G.*

Proof. Let η be the central character of π (Schur) and choose f with $\theta_f = \theta_\pi$ on the elliptic set (elliptic isolation). Then $0 = J(f) - m_{\mathrm{geom}}(\theta_f) = \mathrm{defect}[\pi]$. Finiteness is multiplicity one. □

Theorem 136 (The geometric expansion under (HYP))**.** *(Thesis Thm. 12.2.1 under (HYP).) If (HYP) holds for G, then $J(f) = m_{\mathrm{geom}}(\theta_f)$ for every central character χ and every $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$.*

Proof. Choose the invariant clopen $\Omega \ni 1$ of the expansion near 1; cut $f = f_1 + (f - f_1)$ with f_1 supported in Ω and $\theta_{f_1} = 1_\Omega \theta_f$; apply the expansion near 1 to f_1 and the expansion away from 1 to $f - f_1$, and add, using linearity of J and m_{geom} (Chapter 8). □

Theorem 137 (The induction)**.** *For every GGP triple, the geometric expansion (for every central character) and the multiplicity formula hold.*

Proof. Strong induction on $\dim G$: (HYP) is the induction hypothesis; apply the two preceding results. □

Theorem 138 (Multiplicity formula from the trace formula)**.** *$m(\pi) = m_{\mathrm{geom}}(\theta_\pi)$ (and $m(\pi) < \infty$) for every GGP triple and every tempered π trivial on $A_{G,H}(F)$ (Thesis Thm. 7.2.2 = Thm. 1.0.2).*

Proof. The multiplicity-formula half of the induction. □

Chapter 6

The spectral side

Thesis Ch. 9–10 and Prop. 7.4.1: the spectral expansion of J, in the elliptic form used by the induction, and the vanishing of the defect on properly induced representations.

6.1 Parabolic induction

Definition 139 (Normalized parabolic induction)**.** $i_P^G(\sigma)$: the completion of the smooth $f : G(F) \to H_\sigma$ with $f(pg) = \delta_P(p)^{1/2}\sigma(p)f(g)$ for $\|f\|^2 = \int_K \|f(k)\|^2 dk$, with $G(F)$ acting by right translation (Thesis §3.2).

Definition 140 (Properly induced virtual representations)**.** $R_{\mathrm{ind}}(G,\eta)$: the span of the virtual tempered representations whose character on $G_{\mathrm{reg}}(F)$ is that of some $i_P^G(\sigma)$, P proper, σ tempered with central character η (Thesis §7.4).

Theorem 141 (Van Dijk's formula)**.** *For P proper and σ tempered, $\theta_{i_P^G\sigma}$ vanishes on $G_{\mathrm{ell,reg}}(F)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [23]; [5], (3.4.2)/(3.4.3) = Thesis (3.5.3.2). □

Theorem 142 (Multiplicity of an induced representation)**.** *If a virtual representation Π has the character of $i_P^G(\sigma)$ on $G_{\mathrm{reg}}(F)$, then $m(\Pi) = m(i_P^G\sigma)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: Harish-Chandra (linear independence of characters); [25], §III.4, §VII; [5], §2.2, §2.7. □

Theorem 143 (Vanishing elliptic character means properly induced)**.** *If $\Pi \in R_{\mathrm{temp}}(G,\eta)$ has $\theta_\Pi = 0$ on $G_{\mathrm{ell,reg}}(F)$, then $\Pi \in R_{\mathrm{ind}}(G,\eta)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [3], §6 (orthogonality of elliptic tempered characters); [5], §2.7, (2.7.1). □

Theorem 144 (θ_f is a virtual character on the elliptic set)**.** *For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G,\eta)$ there is a virtual tempered Π with central character η with $\theta_\Pi = \theta_f$ on $G_{\mathrm{ell,reg}}(F)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [5], Prop. 5.6.1(ii), Lemma 5.4.2(i) (= Thesis Prop. 4.8.1), adapted to the χ-equivariant setting and a finite sum. □

6.2 The spectral expansion

Proposition 145 (Thesis Prop. 7.4.1(i) at an induced representation)**.** *Assume (HYP). For $\pi = i_Q^G(\sigma)$, Q proper and σ tempered with central character η, $m(\pi) = m_{\mathrm{geom}}(\theta_\pi)$.*

Proof. Not yet formalized. $m(i_Q^G\sigma) = m(\tilde{\sigma})$ for the GGP triple of the Levi (Thesis Cor. 9.5.1); $m_{\mathrm{geom}}(\theta_{i_Q^G\sigma}) = m_{\mathrm{geom}}^{\tilde{G}}(\theta_{\tilde{\sigma}})$ (Thesis Lemma 7.3.1, [5] Prop. 4.8.1(i)); conclude by (HYP). □

Proposition 146 (The defect vanishes on R_{ind})**.** *Assume (HYP). The defect vanishes on $R_{\mathrm{ind}}(G,\eta)$ (Thesis Prop. 7.4.1(i), extended linearly).*

Proof. On a generator: $m(\Pi) = m(i_Q^G\sigma)$, $m_{\mathrm{geom}}(\Pi) = m_{\mathrm{geom}}(\theta_{i_Q^G\sigma})$, and these agree. □

Proposition 147 (Vanishing elliptic character implies vanishing defect)**.** *Assume (HYP). If $\theta_\Pi = 0$ on $G_{\mathrm{ell,reg}}(F)$ then $m(\Pi) = m_{\mathrm{geom}}(\Pi)$.*

Proof. Such Π lie in R_{ind} (Arthur), where the defect vanishes. □

Theorem 148 (The spectral expansion, elliptic form)**.** *(Thesis Thm. 10.2.1 = Thm. 1.0.4 and Prop. 7.4.1(ii), elliptic form.) Assume (HYP). For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G,\eta)$, $J(f) - m_{\mathrm{geom}}(\theta_f) = \mathrm{defect}(\Pi_f)$, where Π_f is a virtual tempered representation with $\theta_{\Pi_f} = \theta_f$ on $G_{\mathrm{ell,reg}}(F)$.*

Proof. Not yet formalized. The spectral expansion $J(f) = \int_{\mathcal{X}(G,\chi)} D(\pi)\hat{\theta}_f(\pi)m(\bar{\pi})\,d\pi$ (Thesis Thm. 10.2.1, from Thm. 10.1.1, Arthur's local trace formula, the matricial Paley–Wiener theorem, Thm. 9.2.1 and Cor. 9.5.1); on the induced part $m = m_{\mathrm{geom}}$ by Prop. 7.4.1(i). □

Proposition 149 (The spectral expansion in elliptic form)**.** *Assume (HYP). For $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G,\eta)$ there is a virtual tempered Π_f with central character η, $\theta_{\Pi_f} = \theta_f$ on $G_{\mathrm{ell,reg}}(F)$ and $J(f) - m_{\mathrm{geom}}(\theta_f) = m(\Pi_f) - m_{\mathrm{geom}}(\Pi_f)$.*

Proof. Combine the previous theorem with the elliptic virtual character of θ_f. □

Chapter 7

The linear form m_{geom}

Thesis Prop. 12.1.1(i): convergence, linearity and continuity of m_{geom} on quasi-characters.

Lemma 150 (Measurability of the integrand)**.** *For $\theta \in \mathrm{QC}(G,\chi)$, the integrand*

$$t \mapsto D^G(t)^{1/2} c_{\theta,\mathcal{O}_t}(t) \Delta(t)^{s-1/2}$$

of $m_{\mathrm{geom},s}$ is a.e. strongly measurable on $T(F)/A_{G,H}(F)$ (Thesis §6.5).

Proof. The integrand is invariant under $A_{G,H}(F)$ (germs are translation invariant), continuous on the open set $T_\natural$ and 0 off it; so its descent to the quotient is measurable. □

Lemma 151 ($T_\natural$ is open)**.** $T_\natural = \{t \in T : t \in \Gamma(G,H),\ T_t = T\}$ *is open in $T(F)$ (Thesis §6.5).*

Proof. Let $x = t_0 \in T_\natural$, so $T = T_x$, and $t \in T$ near x. The kernel dimension of a continuous family of linear maps is upper semicontinuous. Since $W_x' \subseteq W_t' = \ker(v \mapsto t\iota(v) - \iota(v)t)$, this gives $W_t' = W_x'$. Likewise the moved part $\mathfrak{l}_t$ of $\mathfrak{g}_t$ contains $\mathfrak{l}_x$ (for $Z \in \mathfrak{l}_x$ and small $c \neq 0$, $e^{cZ} \in T_x$ commutes with t), so $\mathfrak{l}_t = \mathfrak{l}_x$ is abelian. The element t is semisimple (elements of T_x are, by the next lemma), so Steinberg's description of centralizers gives $T_t \subseteq T_x$, and $T_x \subseteq T_t$ because T_x is commutative. □

Lemma 152 (Semisimplicity in GSpin)**.** *Over a field of characteristic 0, $x \in \mathrm{GSpin}(V)$ is semisimple (acts semisimply on $C(V)$) if $P(x) \in \mathrm{SO}(V)$ is semisimple.*

Proof. Write $x = s + n$ (Jordan–Chevalley decomposition of left multiplication, $s, n \in K[x]$). Conjugation by x is $c_s + N$ with $c_s = \mathrm{Ad}(s)$ semisimple and N nilpotent, so by uniqueness N is a polynomial in $\mathrm{Ad}(x)$; on $\iota(V)$ it is that polynomial in $P(x)$, nilpotent and semisimple, hence 0. So $s^{-1}x$ is even and commutes with $\iota(V)$, hence is a scalar, and x is a scalar multiple of s. It is used for $t \in T_x$, $x \in \Gamma(G,H)$, whose image $P(t)$ is semisimple ($P(t)$ is scalar on the eigenlines of $P(x)$ and an isometry of the nondegenerate plane E_{-1}). □

Lemma 153 (Continuity on $T_\natural$)**.** *For $\theta \in \mathrm{QC}(G,\chi)$, the integrand of $m_{\mathrm{geom},s}$ is continuous on $T_\natural$.*

Proof. The integrand is the product of $D^G(t)^{1/2} c_{\theta,\mathcal{O}_t}(t)$ (next lemma) and $\Delta(t)^{s-1/2}$. On $T_\natural$, $W_t'' = W_x''$ ($P(t)$ is 1 on W_x' and preserves W_x'', and dimensions agree), so $\Delta(t) = |\det(P(t) - 1)|_{W_x''}|_F$, continuous and nonzero; and $|\cdot|_F$ is locally constant on $F^\times$. □

Lemma 154 (Continuity of the normalized germs)**.** *For $\theta \in \mathrm{QC}(G,\chi)$, $t \mapsto D^G(t)^{1/2} c_{\theta,\mathcal{O}_t}(t)$ is continuous on $T_\natural$.*

Proof. Product of the next two lemmas. □

Lemma 155 (Continuity of D^G on $T_\natural$)**.** *$t \mapsto D^G(t)$ is continuous on $T_\natural$.*

Proof. Fix $x \in T_\natural$ and let $N_t = 1 - \mathrm{Ad}(t)$ on $\mathfrak{g}$, $L = \ker N_x = \mathfrak{g}_x$, $C = \operatorname{im} N_x$. As N_x is semisimple, $\mathfrak{g} = L \oplus C$, and both are N_t-stable for $t \in T$. On $T_\natural$ the centralizers in $G(F)$ of t and x agree (by the centralizer theorem over $\bar{F}$), hence so do the Lie centralizers (exponentiate), so $\ker N_t = L$, $N_t|_C$ is invertible and $D^G(t) = |\det(N_t|_C)|_F$, continuous in t. □

Lemma 156 (Local constancy of germs on $T_\natural$)**.** *For $\theta \in \mathrm{QC}(G,\chi)$, $t \mapsto c_{\theta,\mathcal{O}_t}(t)$ is continuous (locally constant) on $T_\natural$.*

Proof. Near $t_0 \in T_\natural$ write $t = t_0 e^Y$ with $Y = \log(t_0^{-1} t)$ small (a local logarithm from the inverse function theorem). Y lies in $\mathfrak{g}$, is semisimple and central in $\mathfrak{g}_{t_0} = \mathfrak{g}_t$, so $\theta(te^X) = \theta(t_0 e^{X+Y}) = \sum c_{\mathcal{O}} \hat{j}(\mathcal{O}, X+Y) = \sum c_{\mathcal{O}} \hat{j}(\mathcal{O}, X)$: the trace form is nondegenerate on $\mathfrak{g}_{t_0}$, $b(Y,N) = 0$ for nilpotent $N \in \mathfrak{g}_{t_0}$, and the regular set of $\mathfrak{g}_{t_0}$ is open. The expansion at t_0 is one at t, and $\mathcal{O}_t = \mathcal{O}_{t_0}$. □

Lemma 157 (The regular set of $\mathfrak{g}_x$ is open)**.** *For semisimple $x \in G(F)$, the regular semisimple elements of $\mathfrak{g}_x$ form an open subset.*

Proof. Near a regular semisimple Z_0, $\dim \ker(\mathrm{ad}\, Z)^D \le \dim \ker(\mathrm{ad}\, Z_0)^D = \dim \mathfrak{g}_{x,Z_0} = \ell$ (semicontinuity of the rank). For the Jordan decomposition $Z = s + n$ in $\mathfrak{g}_x$, $\mathfrak{g}_{x,s} \subseteq \ker(\mathrm{ad}\, Z)^D$, so $\dim \mathfrak{g}_{x,s} = \ell$; by the rank clauses for GSpin (the pair version of the bound $\dim \mathfrak{so}(V)_N \ge \lfloor \dim V/2 \rfloor$), $\mathfrak{g}_{x,s}$ is then abelian and its only nilpotent element is 0, so $n = 0$ and $Z = s$ is regular semisimple. □

Lemma 158 (Boundedness of germs)**.** *For $\theta \in \mathrm{QC}(G,\chi)$ and a torus T, $D^G(t)^{1/2} c_{\theta,\mathcal{O}_t}(t)$ is bounded on $T_\natural = \{t \in \Gamma(G,H) : T_t = T\}$.*

Proof. Not yet formalized. [5] Prop. 4.5.1 for the individual germs, and compactness of $T(F)/A_{G,H}(F)$. □

Theorem 159 (Integrability of $\Delta^{s-1/2}$)**.** *For $x \in \Gamma(G,H)$, $T = T_x$ and $s > 0$, $t \mapsto 1_{T_\natural}(t) \Delta(t)^{s-1/2}$ is integrable on $T(F)/A_{G,H}(F)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [5], Lemma B.1.2(i), applied as in Thesis Prop. 12.1.1(i). □

Lemma 160 (Thesis (12.1.2), torus by torus)**.** *For $\theta \in \mathrm{QC}(G,\chi)$ and $s > 0$, the integrand of $m_{\mathrm{geom},s}(\theta)$ is integrable on each $T(F)/A_{G,H}(F)$.*

Proof. Bounded germs times the integrable $\Delta^{s-1/2}$. □

Lemma 161 (Existence of the limit)**.** *For $\theta \in \mathrm{QC}(G(F),\chi)$, $\lim_{s \to 0^+} m_{\mathrm{geom},s}(\theta)$ exists (Thesis Prop. 12.1.1(i), (12.1.4)).*

Proof. Not yet formalized. Induction on $\dim G$: semisimple descent away from 1, descent to the Lie algebra and homogeneity near 1. □

Proposition 162 (m_{geom} converges)**.** *For θ in the span $\mathrm{QC}_{\mathrm{fin}}$ of the spaces $\mathrm{QC}(G,\chi)$, the limit $m_{\mathrm{geom}}(\theta)$ exists (Thesis Prop. 12.1.1(i)).*

Proof. Linearity of $m_{\mathrm{geom},s}$ for $s>0$ and the limit for each χ. □

Proposition 163 (m_{geom} is additive)**.** $m_{\mathrm{geom}}(\theta_1+\theta_2)=m_{\mathrm{geom}}(\theta_1)+m_{\mathrm{geom}}(\theta_2)$ *on* $\mathrm{QC}_{\mathrm{fin}}$ *(Thesis Prop. 12.1.1(i)).*

Proof. From convergence. □

Theorem 164 (Harish-Chandra's compactness lemma)**.** *Every compact $C\subset G(F)$ is contained in an open, closed, invariant, $A_G(F)$-stable Ω that is compact modulo conjugation and $A_G(F)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [10], Part I; [5], Prop. 3.1.1(i) (= Thesis Prop. 3.5.1(i)), adapted. □

Proposition 165 (Support of m_{geom})**.** *There is an invariant clopen Ω, compact modulo conjugation and $A_G(F)$, such that $m_{\mathrm{geom}}(\theta)=0$ for every $\theta\in\mathrm{QC}(G,\chi)$ vanishing on $\Omega\cap G_{\mathrm{reg}}(F)$ (Thesis Prop. 12.1.1(i), continuity).*

Proof. $\Gamma(G,H)$ is a finite union of images of tori compact modulo $A_{G,H}(F)$; take Ω around them; germs only depend on θ near the point. □

Chapter 8

The geometric expansion

Thesis §12.2–12.5: the geometric expansion $J(f) = m_{\mathrm{geom}}(\theta_f)$ under (HYP), split into a neighbourhood of the centre (descent to the Lie algebra) and its complement (semisimple descent).

8.1 Extending J to quasi-characters

Definition 166 (Extensions J_{qc} of J)**.** A linear form J_{qc} on $\mathrm{QC}(G, \chi^{-1})$ that depends only on values on $G_{\mathrm{reg}}(F)$, satisfies $J(f) = J_{\mathrm{qc}}(\theta_f)$, is continuous, and $J_{\mathrm{qc}} - m_{\mathrm{geom}}$ is supported in the elliptic set (Thesis Prop. 12.2.3). This differs from Thesis Prop. 12.2.3, which has $\mathrm{Supp}(J_{\mathrm{qc}}) \subset G(F)_{\mathrm{ell}}$: here $\Gamma(G, H)$ need not lie in the elliptic set.

Lemma 167 ($J - m_{\mathrm{geom}}$ only sees the elliptic set)**.** *Assume (HYP). If $\theta_f = \theta_{f'}$ on $G_{\mathrm{ell,reg}}(F)$ then $J(f) - m_{\mathrm{geom}}(\theta_f) = J(f') - m_{\mathrm{geom}}(\theta_{f'})$ (Thesis Prop. 7.4.1(ii)).*

Proof. Both are defects of virtual representations with the same elliptic character. □

Proposition 168 (Existence of J_{qc})**.** *Assume (HYP). Then J extends to some J_{qc} as above (Thesis Prop. 12.2.3).*

Proof. $J_{\mathrm{qc}}(\theta) = m_{\mathrm{geom}}(\theta) + (J(f) - m_{\mathrm{geom}}(\theta_f))$ for any f with $\theta_f = \theta$ on the elliptic set (elliptic isolation); well defined by the previous lemma. □

Lemma 169 ($J(f)$ depends only on θ_f)**.** *Assume (HYP). If $\theta_f = \theta_{f'}$ on $G_{\mathrm{reg}}(F)$ then $J(f) = J(f')$ (Thesis Prop. 12.2.3).*

Proof. $J = J_{\mathrm{qc}} \circ \theta$. □

8.2 Away from the centre

Thesis §12.3.

Lemma 170 (Invariant clopen sets are saturated along unipotent radicals)**.** *For an invariant clopen Ω, $p \in P(F)$ and $u \in U(F)$, $pu \in \Omega \iff p \in \Omega$.*

Proof. Levi decomposition $P = MU$ in GSpin and a cocharacter of the split centre of M, lifted to GSpin, contracting U (Thesis §5.1.2): $a_k C_k(pu)C_k^{-1}$ and $a_k C_k p C_k^{-1}$ ($a_k \in A_G(F)$) have the same limit, which lies in Ω iff p (or pu) does, as Ω is open and closed. □

Proposition 171 (Partition of unity)**.** *Assume (HYP). If invariant clopen sets Ω_i are each either good (the geometric expansion holds for f supported there) or contained in the zero set of θ, and θ_f is supported in $\bigcup_i \Omega_i$ and in the support of θ, then $J(f) = m_{\mathrm{geom}}(\theta_f)$ (Thesis Prop. 12.3.1, [5] §11.6).*

Proof. Induction on the number of sets, cutting f by Ω_i. □

Theorem 172 (Neighbourhoods of the near-central set)**.** *For every neighbourhood U of 1 there is an open, closed, invariant, $A_G(F)$-stable Ω containing every near-central element (1 in the closure of $A_G(F)c^{G(F)}$), each element of which is conjugate into U modulo $A_G(F)$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [10], Part I; [5], §3.3 (= Thesis §3.5.2), adapted. □

Lemma 173 (Thesis Prop. 12.3.1, local form)**.** *Assume (HYP). Every c whose semisimple part is not in $A_G(F)$ has an invariant clopen neighbourhood that either contains no elliptic regular element or on which the geometric expansion holds.*

Proof. Not yet formalized. Semisimple descent to $G_x = G'_x \cdot G''_x$: (HYP) for the smaller GGP triple (G'_x, H'_x, ξ'_x) and Arthur's local trace formula on G''_x (Thesis (12.3.3), Thm. 4.7.1). □

Proposition 174 (Thesis Prop. 12.3.1)**.** *Assume (HYP). If $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$ and θ_f vanishes near 1, then $J(f) = m_{\mathrm{geom}}(\theta_f)$.*

Proof. Cover the support of θ_f (compact modulo conjugation and A_G) by finitely many good sets and glue. □

8.3 Near the centre: descent to the Lie algebra

Thesis §12.4–12.5 and Ch. 11. Here $\mathfrak{g}^0 = \mathfrak{so}(W) \oplus \mathfrak{so}(V)$.

Definition 175 (The distribution J^{Lie})**.** $J^{\mathrm{Lie}}(f) = \int_{H(F)\backslash G(F)/A_G(F)} K^{\mathrm{Lie}}(f,x)\,dx$ for f on $\mathfrak{g}^0(F)$ (Thesis §10.1; [17] §4, [5] §8.1).

Definition 176 (θ_f on the Lie algebra)**.** For $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}^0(F))$, the quasi-character θ_f (Thesis §4.8; [5] §5.5).

Definition 177 ($m^{\mathrm{Lie}}_{\mathrm{geom}}$)**.** $m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta) = \lim_{s\to 0^+} m^{\mathrm{Lie}}_{\mathrm{geom},s}(\theta)$ (Thesis Prop. 12.1.1(iii)).

Lemma 178 (Descent to the Lie algebra near 1)**.** *(Thesis Prop. 12.4.1(i), Prop. 12.1.1(iv).) If neither W nor V is a hyperbolic plane, there is an invariant clopen $\Omega \ni 1$ such that every $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$ supported in Ω has a strongly cuspidal F on $\mathfrak{g}^0$ with $J(f) = J^{\mathrm{Lie}}(F)$ and $m_{\mathrm{geom}}(\theta_f) = m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_F)$.*

Proof. Not yet formalized. $\Omega = A_G(F)\exp(\omega)$ for a $G(F)$-excellent ω, $F = (j^H_G)^{1/2} f_\omega$. □

Theorem 179 (Spectral expansion of J^{Lie})**.** *(Thesis Thm. 11.6.1, in normalization-free form.) If neither W nor V is a hyperbolic plane, $J^{\mathrm{Lie}}(f)$ depends only on $\theta_f|_{\mathfrak{g}^0_{\mathrm{reg}}}$.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [17], Thm. 6.6.1; [26], §9–10; following [5], Thm. 10.8.1. □

Lemma 180 (J^{Lie} factors through θ_f)**.** *If neither W nor V is a hyperbolic plane, $J^{\mathrm{Lie}}(f) = J^{\mathrm{Lie}}_{\mathrm{qc}}(\theta_f)$ for every strongly cuspidal f on $\mathfrak{g}^0$ (Thesis Prop. 12.4.1(ii)).*

Proof. From the spectral expansion of J^{Lie}. □

Lemma 181 (Thesis Prop. 12.5.1, test-function form)**.** *Assume (HYP), and that J^{Lie} factors through θ_f by a form seeing only $\mathfrak{g}^0_{\mathrm{reg}}$. Then $J^{\mathrm{Lie}}(f) - m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_f) = \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}^0)} c_{\mathcal{O}}\, c_{\theta_f,\mathcal{O}}(0)$ for some constants $c_{\mathcal{O}}$.*

Proof. Not yet formalized. The difference is a homogeneous continuous form, vanishing on quasi-characters with $0 \notin \mathrm{Supp}\theta$ by semisimple descent and (HYP) (Prop. 12.4.1(iv)); such a form is a combination of germs at 0 ([5] Prop. 4.6.1). □

Proposition 182 (Thesis Prop. 12.5.1)**.** *Assume (HYP), neither W nor V a hyperbolic plane. Then $J^{\mathrm{Lie}}(f) - m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_f) = \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}^0)} c_{\mathcal{O}}\, c_{\theta_f,\mathcal{O}}(0)$.*

Proof. The test-function form with $J^{\mathrm{Lie}}_{\mathrm{qc}}$. □

Theorem 183 (The endgame)**.** *(Thesis §12.5.) If $J^{\mathrm{Lie}}(f) - m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_f) = \sum_{\mathcal{O}} c_{\mathcal{O}} c_{\theta_f,\mathcal{O}}(0)$ for every strongly cuspidal f on $\mathfrak{g}^0$, then every $c_{\mathcal{O}}$ vanishes.*

Proof. Not verbatim in the literature (adapted from, or derived from, published results); not yet formalized. Ingredients: [26], §11.4–11.6; [17], Prop. 7.3.7, Lemma B.3.1. □

Theorem 184 ($J^{\mathrm{Lie}} = J^{\mathrm{Lie}}_{\mathrm{geom}}$)**.** *Assume (HYP), neither W nor V a hyperbolic plane. Then $J^{\mathrm{Lie}}(f) = m^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_f)$ for strongly cuspidal f on $\mathfrak{g}^0$ (Thesis Thm. 12.2.2).*

Proof. Prop. 12.5.1 and the endgame. □

Lemma 185 (The case V a hyperbolic plane)**.** *If V is a hyperbolic plane ($\dim W = 1$, $r = 0$), the geometric expansion holds on all of $G(F)$.*

Proof. G is a split torus, $A_G = G$; then $J(f) = f(1)$, $\theta_f = f$ and $m_{\mathrm{geom}}(\theta_f) = c_{\theta_f,0}(1) = f(1)$, using a self-dual measure for $\hat{j}(0,\cdot) = 1$. □

Lemma 186 (The case W a hyperbolic plane)**.** *Assume (HYP) and W a hyperbolic plane. There is an invariant clopen $\Omega \ni 1$ on which the geometric expansion holds.*

Proof. Not yet formalized. Here $A_{\bar{G}} = \mathrm{SO}(W)$ is a nontrivial split central torus; the Lie-algebra argument must be run modulo $A_{\bar{G}}$, which the cited literature does not state in this form ([5] Ch. 10–11 modulo $A_{\bar{G}}$). □

Theorem 187 (The geometric expansion near the centre)**.** *(Thesis Prop. 12.4.1 and Thm. 12.2.2.) Assume (HYP). For every χ there is an invariant clopen $\Omega \ni 1$ with $J(f) = m_{\mathrm{geom}}(\theta_f)$ for every $f \in \mathcal{C}_{\mathrm{scusp}}(A_G\backslash G, \chi^{-1})$ supported in Ω.*

Proof. Cases: V a hyperbolic plane; W a hyperbolic plane; otherwise descend to $\mathfrak{g}^0$ and use $J^{\mathrm{Lie}} = J^{\mathrm{Lie}}_{\mathrm{geom}}$. □

Chapter 9

Further statements not yet used by the main theorem

These statements are formalized but not (yet) on the path of the main theorem: the analytic estimates of Thesis Ch. 8 and 10 (the convergence of J is stated separately above), the Levi decomposition and transport of m_{geom} (Thesis §7.2, §9.3), and checks of the normalization of $\hat{j}$ on $\mathfrak{sl}_2$.

9.1 Estimates

Thesis Ch. 8, §10.1.

Proposition 188 (Good minimal parabolic subgroups)**.** *(Thesis Prop. 8.1.1(i), existence.) There are full isotropic flags of W and V whose stabilizer $\bar{P}$ satisfies $H \cap \bar{U}_{\min} = \{1\}$, with $H(F)\bar{P}(F)$ open in $G(F)$ and of Haar-null complement.*

Proof. Not yet formalized (adapted from [5] Prop. 6.4.1). "Good" is replaced by its two consequences used in the estimates. □

Lemma 189 (Thesis Lemma 8.2.1(i))**.** *For $r = 0$ there is $\epsilon > 0$ with $\int_{H_0(F)/A_{G,H}(F)} \Xi^{G_0}(h_0) e^{\epsilon\sigma(h_0)}\, dh_0 < \infty$. This differs from Thesis Lemma 8.2.1(i), which integrates over $H_0(F)/A_{H_0}(F)$.*

Proof. Not yet formalized (adapted from [5] Lemma 6.5.1(i)). □

Lemma 190 (Thesis Lemma 8.2.1(ii))**.** *There is $d > 0$ with $\int_{H(F)/A_{G,H}(F)} \Xi^{G}(h)\sigma(h)^{-d}\, dh < \infty$.*

Proof. Not yet formalized (adapted from [5] Lemma 6.5.1(ii)). □

Lemma 191 (Thesis Lemma 8.2.1(iii))**.** *For every $\delta > 0$ there is $\epsilon > 0$ with $\int_{H(F)/A_{G,H}(F)} \Xi^{G}(h) e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta}\, dh < \infty$.*

Proof. Not yet formalized (adapted from [5] Lemma 6.5.1(iii)). □

Theorem 192 (Thesis Thm. 10.1.1(i)). *There are $d > 0$ and a continuous seminorm ν on $\mathcal{C}(A_G(F)\backslash G(F), \chi^{-1})$ with $|K(f,x)| \le \nu(f)\,\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^d$.*

Proof. Not yet formalized (adapted from [5] Thm. 8.1.1(i)). □

9.2 Levi subgroups

Proposition 193 (The Levi decomposition). *For a GGP datum $\tilde{d}$ realizing the Levi pair $(\tilde{W}, \tilde{V})$ of a pair of isotropic flags, there is a surjective homomorphism with central kernel from the Levi quotient P/U onto $\prod \mathrm{GL}(Z'_{i,W}) \times \prod \mathrm{GL}(Z'_{j,V}) \times \tilde{G}(F)$ (Thesis §5.1.2, (9.3.1)–(9.3.3)).*

Proof. Not yet formalized: it needs the Clifford-algebra description of the Levi subgroup. □

Lemma 194 (Transport of m_{geom}). *For an isomorphism of GGP triples $e : G \to G'$ and a quasi-character θ' of $G'(F)$, $m_{\mathrm{geom}}(\theta' \circ e) = m_{\mathrm{geom}}(\theta')$ (Thesis §7.2, Prop. 12.1.1).*

Proof. Not yet formalized: needs the transport of $\mathcal{O}_x$ and $\hat{j}$. □

9.3 Normalization checks on $\mathfrak{sl}_2$

Sanity checks of the definition of $\hat{j}$ for $\mathrm{GL}_2(F)$ acting on $\mathfrak{sl}_2(F)$ with the trace form.

Lemma 195 (The regular orbit measure exists). *The $\mathrm{GL}_2(F)$-invariant measure on the regular nilpotent orbit, normalized by the self-dual measure on $\mathfrak{g}/\mathfrak{g}_E$ ([5] §1.6, Thesis §2.4), exists.*

Proof. Not yet formalized: needs the p-adic inverse function theorem and invariant measures on homogeneous spaces. □

Lemma 196 ($\hat{j}(\mathcal{O}_{\mathrm{reg}}, \cdot)$ on the split torus). *For $X = x\,\mathrm{diag}(1,-1)$, $x \neq 0$: $\hat{j}(\mathcal{O}_{\mathrm{reg}}, X) = 2/|2x|$.*

Proof. Not yet formalized (hand computation, every residue characteristic). □

Lemma 197 ($\hat{j}(\mathcal{O}_{\mathrm{reg}}, \cdot)$ on elliptic tori). *For ε a nonsquare and $x \neq 0$, $\hat{j}(\mathcal{O}_{\mathrm{reg}}, x\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}) = 0$.*

Proof. Not yet formalized: $\mu_{\mathcal{O}_{\mathrm{reg}}}$ is induced from the Borel, so its Fourier transform is induced from the split torus. □

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