Constuctible Numbers and Regular n-gons

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1 Introduction

Using algebraic insights we are able to answer several geometric questions that have been asked since the times of the Greeks. The Greeks were very fascinated by geometry and one area of that is classical straightedge and compass constructions. A compass is used to measure and translate arbitrary distances and is often used for drawing circles of a given radius. A straightedge is used simply for connecting any two points with a straight line. All straightedge and compass constructions are composed of the following four basic operations:

1. connecting two given points by a line
2. findind a point of intersection of two straight lines
3. drawing a circle with given radius and center
4. finding the points of intersection of a straight line and a circle or the intersection of two circles

By using only these operations the Greeks posed the following questions:

1. Is it possible to construct a cube with precisely twice the volume of a given cube?

2. Is it possible to trisect any given angle?

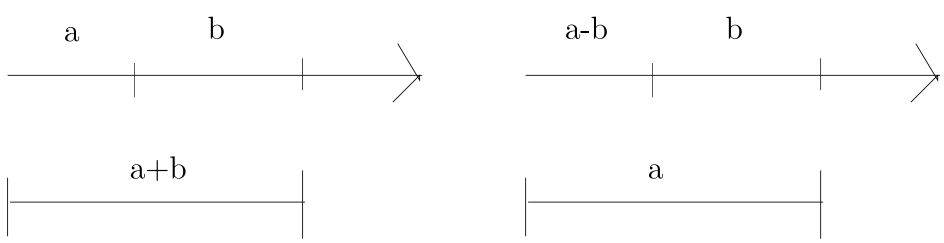
3. Is it possible to construct a square whose area is precisely the area of a given circle?

In addition they wanted to know how to construct as many regular polygons as they could and it wasn't until much later that Gauss proved that a regular n-sided polygon can only be constructed with straightedge and compass if the odd prime factors of n are distinct Fermat primes

2 Constructible Numbers

Definition. *Constructible numbers are those whose length can be constructed from a fixed unit distance using straightedge and compass constructions.*

Given two lengths a, b it is clear to see that you can construct lengths $a \pm b$.



(a) Constructing $a + b$

(b) constructing $a - b$

Figure 1: Constructing $a \pm b$

By constructing parallel lines and using similar triangles we are able to also construct lengths ab and $\frac{a}{b}$ as shown below:

Figure 2: First constructing triangles

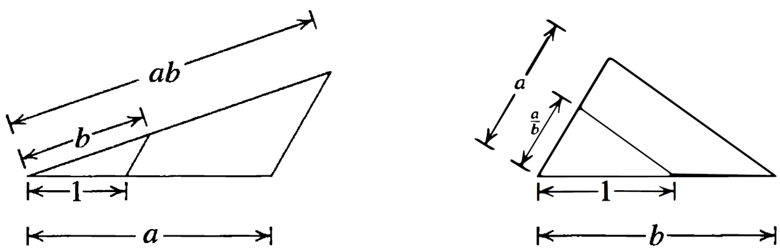


Figure 3: Then drawing parallel lines to construct similar triangles with side lengths ab and $\frac{a}{b}$

This means that all constructible numbers are closed under field operations, making the set of constructible numbers a field. Thus by starting with a unit length, 1, we can construct all of $\mathbb{Q}$. But in addition to these constructions we can also construct square roots. This can be done by constructing a circle intersecting $(-a, 0)$ and $(1, 0)$ with its center at $(\frac{1-a}{2}, 0)$ the circle will then intersect the y-axis at $(0, \sqrt{a})$.

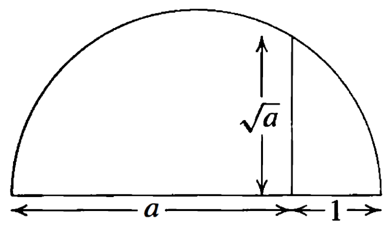


Figure 4: Constructing $\sqrt{a}$

This is easy to show as the equation of the circle give us:

$$\begin{aligned}(x-\tfrac{1-a}{2})^2+y^2&=(\tfrac{a+1}{2})^2\\ \tfrac{a^2-2a+1}{4}+y^2&=\tfrac{a^2+2a+1}{4}\quad\text{when } x=0\\ y^2&=a\\ y&=\sqrt{a}\end{aligned}$$

This means that the field we can construct is larger that $\mathbb{Q}$. By the above construction we can obviously construct fields whose extension is of degree 2^k but we have not shown that those are the only constructible field extensions of $\mathbb{Q}$.

Theorem. *If the element $\alpha \in \mathbb{R}$ is obtained from a field $F \subset \mathbb{R}$ by a series of straightedge constructions then $[F(\alpha) : F] = 2^k$ for some integer $k \geq 0$.*

Proof. By using construction types (1) and (2) we can intersect two lines with equations $y = mx + b$ and $y = m'x + b'$ with $m, m', b, b' \in F$. Solving for x gives $x = \frac{b'-b}{m-m'}$ which still lies in F. By constructions (3) and (4) we can intersect a circle with a line or another circle. The intersection of $(x-h)^2+(y-k)^2 = r^2$ and $y = mx+b$ is the solution of $(x-h)^2+(mx+b-k)^2 = r^2$ which is still at worst a quadratic extension. For the intersection of two circles, $(x-h)^2 + (y-k)^2 = r^2$ and $(x-h')^2 + (y-k')^2 = r'^2$ we can show by subtracting the second equation from the first that this is the same as the intersection of $(x-h)^2+(y-k)^2=r^2$ and $2(h'-h)x+2(k'-k)y = r^2-h^2-k^2-r'^2+h'^2+k'^2$ which has reduced to the previous case of intersecting a line with a circle. So this means that we can only construct quadratic extensions with a straightedge and compass. $\square$

3 Answers to the Classical Greek Problems

The above proposition is sufficient to prove that none of the constructions posed by the Greeks are possible to construct.

1. Doubling the volume of a unit cube requires constructing side lengths of $\sqrt[3]{2}$ and since $[F(\sqrt[3]{2}):F]=3\neq 2^k$ this cannot be constructed.

2. Trisecting a given angle is equivalent to constructing $\cos(\theta/3)$ and by the triple angle formula we have that $\cos\theta = 4\cos^3(\theta/3)-3\cos(\theta/3)$. Considering $\theta=60$ and $\alpha = 2\cos 20$ we have that α is the real root of the cubic polynomial $\alpha^3-3\alpha-1=0$. This polynomial is irreducible as it has no rational roots from the rational roots test so it has no real roots and thus α is not constructible so not all angles are trisectable.

3. Squaring the cirlce is equivalent to trying to construct π. But as π is transcendental over $\mathbb{Q}$ we have that $[\mathbb{Q}(\pi):\mathbb{Q}]$ is not even finite so not a power of 2. Thus you cannot square the circle with straightedge and compass.

4 Constructible Regular N-gons

Definition. *A complex number is constructible if its real and imaginary components can be constructed.*

All of our constructions have been taking place in $\mathbb{R}^2$ which is isomorphic to $\mathbb{C}$ so it is very simple to extend our definitions of constructible numbers to $\mathbb{C}$. If $a, b \in \mathbb{R}$ and are constructible then we can obviously construct (a, b) so we can say that $a + bi$ is constructible.

Thus we can say constructing a regular n-gon is equivalent to constructing n^{th} roots of unity as the n^{th} roots of unity make up the vertices of a regular n-gon on the unit circle with a vertex at the point (0,1). Since we've established that the only constructible number are those that are in a field with degree extension 2^k we can construct n-gons where $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = 2^k$. Dummit and Foote shows that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$ so the problems reduces to when $\varphi(n) = 2^k$. Where $\varphi(n)$ is defined to be the number of integers less that n that are coprime to n

Theorem. $\varphi(n) = \varphi(n) = n \prod_{p|n}(1 - \frac{1}{p})$

Proof. $\varphi(p) = p - 1$ as all integers less than a prime are coprime to it. For $n = p^k$ all integers less than or equal to it are coprime besides those which are a multple of p. There are p^{k-1} multiples of p less than or equal to p^k so $\varphi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1) = p^k(1 - \frac{1}{p})$. By the chinese remainder theorem we have that if a, b are coprime then $\gcd(n, ab) = 1$ iff $\gcd(n, a) = 1$ and $\gcd(n, b) = 1$. This implies that if a, b are coprime then $\varphi(ab) = \varphi(a)\varphi(b)$. Since every number can be decomposed as $n = p_1^{k_1} p_2^{k_2} \dots$

and all the $p_i^{k_i}$ are coprime to eachother we have that for any n, $\varphi(n) = \prod_{p|n} p^k(1-\frac{1}{p}) = n\prod_{p|n}(1-\frac{1}{p})$ $\square$

Considering when $n = p^k$, $\varphi(n) = p^{k-1}(p-1)$. We can solve for the values of n of this form that are constructible by setting $2^m = p^{k-1}(p-1)$. If $k = 1$ then $p = 2^m + 1$ and if $k > 1$ then we have that $p = 2$. Since we also have that if two number a, b are relatively prime then $\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(n) = 2^k$ only for any number n that that is a composite of primes of the form $2^m + 1$ and 2^l.

This composition fact means that we can compose any two constructible m and n gons to make an mn-gon. It is very easy to actually compose any two n-gons that you are able to construct. By angle bisection you can very easily double an n-gon:

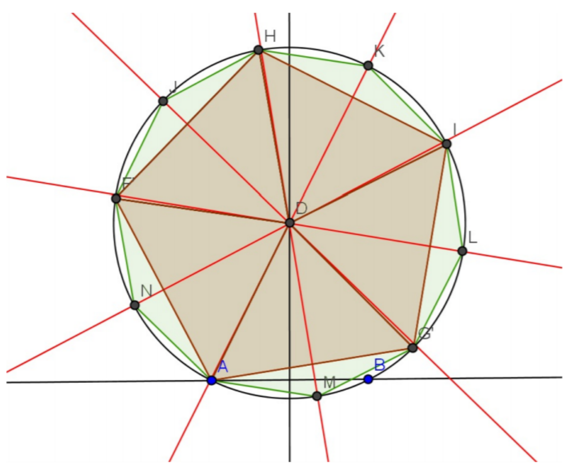


Figure 5: Constructing a regular Decagon from a regular Pentagon

and by overlaying a regular m-gons and a regular n-gon with the same center and sharing a vertex it is easy to construct a regular mn-gon:

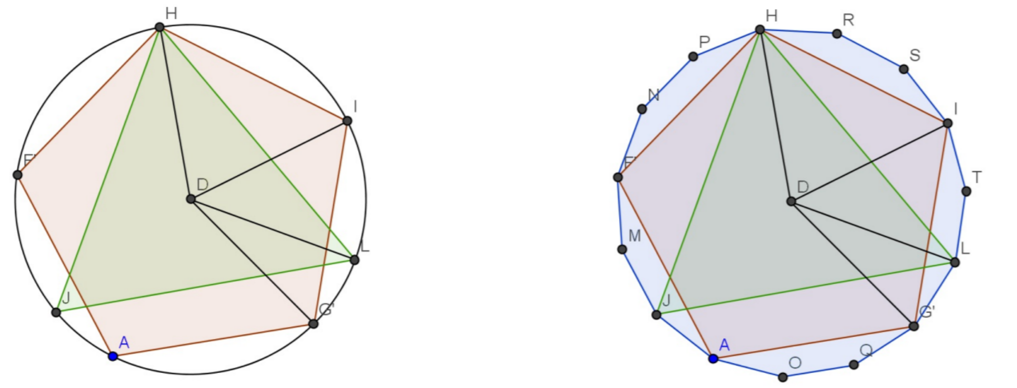


Figure 6: Constructing a regular 15-gon from a regular Pentagon and Triangle

Definition. *A Fermat Prime is a prime number of the form* $p = 2^{2^k} + 1$

Next we can show that any prime of the form $p = 2^m + 1$ is in fact a fermat prime. This can be shown by the fact that $x+1|x^a+1$ for odd a which follows from the geometric sum formula: $\frac{x^a+1}{x+1} = \frac{(-x)^a-1}{(-x)-a} = 1-x+x^2-\cdots+(-x)^{a-1}$. This means any $p = 2^m + 1 = 2^{a2^k} + 1$ with a odd is divisible by $2^{2^k} + 1$. So for p to be prime and indivisible $a = 0$ so p must be of the fowm $p = 2^{2^k} + 1$.

Thus we have now proven the Gauss-Wantzel theorem:

Theorem. *A regular n-gon is constructible if and only if n is of the form $n = 2^k p_1 p_2 \dots p_i$ where $k \geq 0$ and $p_1, p_2, \dots, p_i$ are Fermat Primes.*

5 Constructing Prime n-gons

As we've seen above by composition we can construct any composite regular m-gon from the regular prime n-gons where $n|m$, so to actually construct any constructible n-gons we just need to construct these prime n-gons. Although

Guass proved the constructibility of many regular n-gons their constructions were not found for many years to come.

5.1 Carlyle Circles

A useful tool for accomplishing this are Carlyle circles. Carlyle circles are circles associated to a specific quadratic equation such that the roots of the quadratic are the horizontal components of the intersection of the circle with the x-axis. Give a quadratic $x^2 - sx + p = 0$ we can construct its Carlyle circle by drawing the circle with diameter between the points $(0, 1)$ and (s, p)

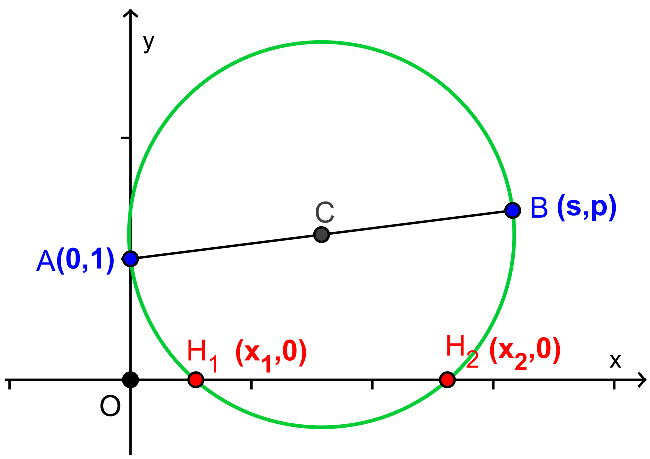


Figure 7: Carlyle Circle

This circle has center $(\frac{s}{2}, \frac{p+1}{2})$ and radius $\sqrt{(\frac{s}{2})^2 + (\frac{p-1}{2})^2}$. This gives us a circle equation:

$$\begin{aligned}(x - \tfrac{s}{2})^2 + (y - \tfrac{p+1}{2})^2 &= (\tfrac{s}{2})^2 + (\tfrac{p-1}{2})^2 \\ x^2 - sx + \tfrac{s^2}{4} + \tfrac{p^2+2p+1}{4} &= \tfrac{s^2}{4} + \tfrac{p^2-2p+1}{4} \quad \text{when } y = 0 \\ x^2 - sx + p &= 0\end{aligned}$$

This gives us an easy way to create specific quadratic field extensions. For any specific quadratic equation we can construct this circle using its coefficients to find its roots.

While this gives a general method to construct the roots of any quadratic and thus construct any field we need this does not provide a general method for constructing very large n-gons. The 65537-gon involves finding the roots of $x^2 + x - 2^{14} = 0$ which would be very difficult to physically do. [1]

5.2 Constructing the Regular 17-gon

First we will define the following where ζ is the first principal 17th root of unity:

$$\eta_1 = \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16}$$
$$\eta_2 = \zeta^3 + \zeta^5 + \zeta^6 + \zeta^7 + \zeta^{10} + \zeta^{11} + \zeta^{12} + \zeta^{14}$$
$$\eta_1' = \zeta + \zeta^4 + \zeta^{13} + \zeta^{16}$$
$$\eta_2' = \zeta^2 + \zeta^8 + \zeta^9 + \zeta^{15}$$
$$\eta_3' = \zeta^6 + \zeta^7 + \zeta^{10} + \zeta^{11}$$
$$\eta_4' = \zeta^3 + \zeta^5 + \zeta^{12} + \zeta^{14}$$
$$\eta_1'' = \zeta + \zeta^{16}$$
$$\eta_2'' = \zeta^4 + \zeta^{13}$$

We claim that $\eta_i \in \mathbb{R}$ for all i. This can be seen as every η is the sum of terms of the form $\zeta^k + \zeta^{17-k}$ with different k values. ζ^k and ζ^{17-k} are complex numbers with the same real component and opposite imaginary components so their sum is real, specifically: $\zeta^k + \zeta^{17-k} = 2\cos(\frac{2\pi k}{17})$. This means that all of these values correspond to lengths which we will see can be constructed.

We've seen that the sum of all nth roots of unity is 0 so the sum of all roots minus the root e^0 is -1, which means that $\eta_1 + \eta_2 = -1$. If we rewrite all of these roots in the form of $\zeta^k = e^{\frac{2i\pi k}{17}}$ then $\eta_1 = e^{\frac{2i\pi}{17}} + e^{\frac{4i\pi}{17}} + e^{\frac{8i\pi}{17}} + e^{\frac{16i\pi}{17}} + e^{\frac{18i\pi}{17}} + e^{\frac{26i\pi}{17}} + e^{\frac{30i\pi}{17}} + e^{\frac{32i\pi}{17}}$ and $\eta_2 = e^{\frac{6i\pi}{17}} + e^{\frac{10i\pi}{17}} + e^{\frac{12i\pi}{17}} + e^{\frac{14i\pi}{17}} + e^{\frac{20i\pi}{17}} + e^{\frac{22i\pi}{17}} + e^{\frac{24i\pi}{17}} + e^{\frac{28i\pi}{17}}$. Since $e^{\frac{2i\pi k}{17}} = e^{\frac{2i\pi(k+17)}{17}}$ we have that $\eta_1\eta_2 = 4(e^{\frac{2i\pi}{17}} + e^{\frac{4i\pi}{17}} + e^{\frac{8i\pi}{17}} + e^{\frac{16i\pi}{17}} + e^{\frac{18i\pi}{17}} + e^{\frac{26i\pi}{17}} + e^{\frac{30i\pi}{17}} + e^{\frac{32i\pi}{17}} + e^{\frac{6i\pi}{17}} + e^{\frac{10i\pi}{17}} + e^{\frac{12i\pi}{17}} + e^{\frac{14i\pi}{17}} + e^{\frac{20i\pi}{17}} + e^{\frac{22i\pi}{17}} + e^{\frac{24i\pi}{17}} + e^{\frac{28i\pi}{17}}) = 4(-1) = -4$ by the same logic as above. This means that η_1 and η_2 are the roots of $x^2 + x - 4 = 0$ so they can be constructed by drawing the corresponding Carlyle circle.

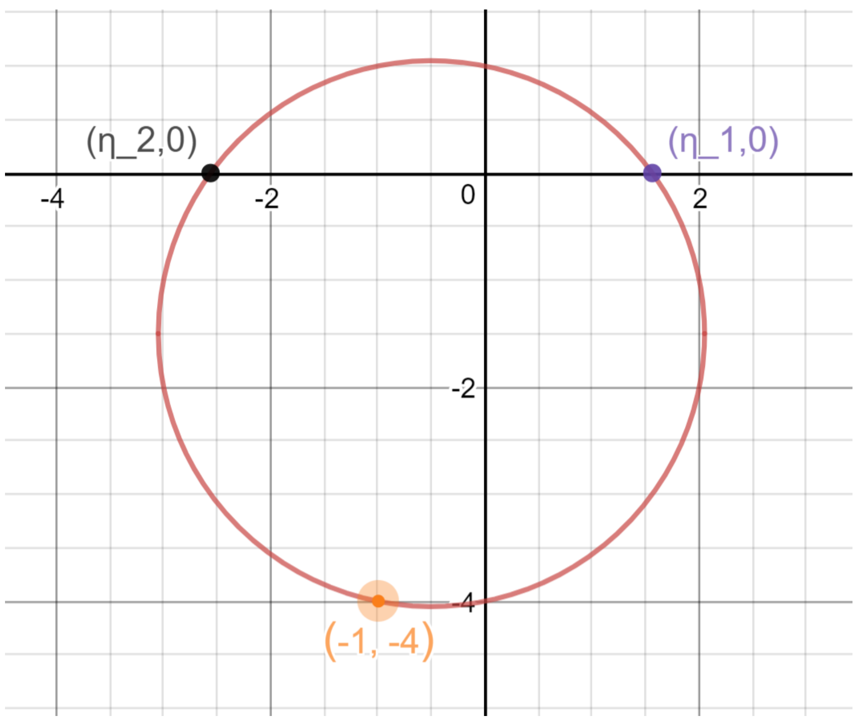


Figure 8: Constructing η_1 and η_2

By similar computation it is simple to show that η_1' and η_2' are roots of the equation $x^2 - \eta_1 - 1 = 0$ and η_4' and η_4' are roots of the equation $x^2 - \eta_2 - 1 = 0$

and thus are constructible. Then we can show that η''_1 and η''_2 are the roots of $x^2 - \eta'_1 x + \eta'_4$. And thus are also easily constructed using Carlyle circles.

After constructing these lengths through many Carlyle circles it is very easy to construct the 17-gon. First construct the point $(0, \frac{\eta''_1}{2})$. This is the real component of ζ as $\frac{\eta''_1}{2} = \frac{e^{\frac{2i\pi}{17}} + e^{\frac{32i\pi}{17}}}{2} = \frac{\cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17} + \cos\frac{32\pi}{17} + i\sin\frac{32\pi}{17}}{2} = \frac{2\cos\frac{2\pi}{17}}{2} = \cos\frac{2\pi}{17}$. This means that we just need to construct a circle of radius 1 at the origin and find its intersection with the vertical line passing through $(0, \frac{\eta''_1}{2})$ to find ζ.

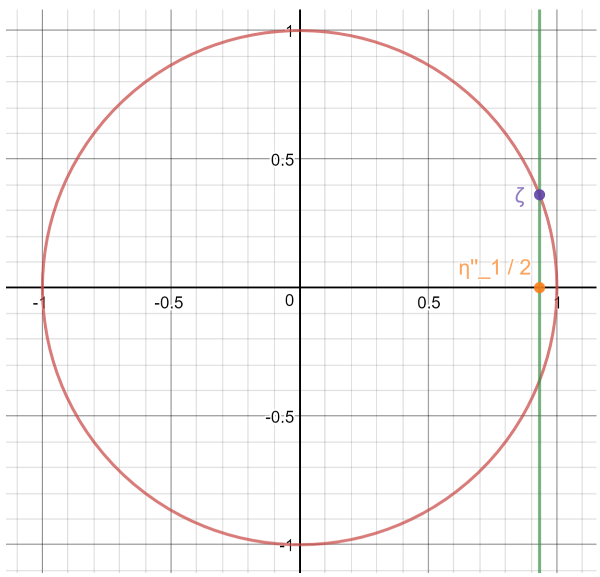


Figure 9: Constructing ζ

Once we have found ζ we just need to mark off the length between it and $(0, 1)$ and then draw each of the other n^{th} roots of unity as they're evenly spaced along the circle.

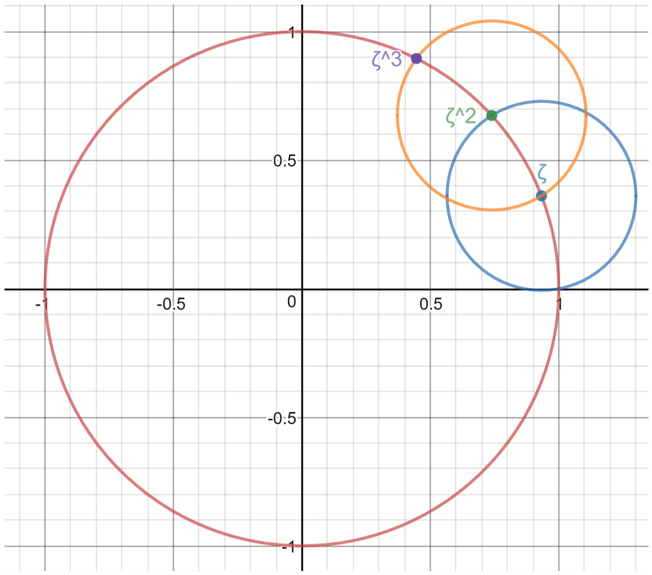


Figure 10: Constructing all ζs

Then all we need to do is draw the lines between each ζ to construct the regular 17-gon.

6 Conclusion

This is another example of how it required advances in more theoretical math was required before a simple question could be answered. Using field theory we can much better understand the Greek question of what is constructible and what isn't. The methods and figures in this paper were taken from Dummit and Foote in *Abstract Algebra* [2] and Kuh in *Constructible regular n-gons*[3]

References

[1] Duane W. DeTemple. Carlyle circles and the lemoine simplicity of polygon constructions. *Am. Math. Monthly*, 98(2):97–108, January 1991.

[2] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2004.

[3] Devin Kuh. Constructible regular n-gons. 2013.