Computing Values of Symmetric Square L-Functions using Ichino's Pullback Formula

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1 Introduction

The goal of this project was to compute the constants in Ichino's pullback formula for Saito-Kurokawa lifts which correspond to specific values of symmetric square L-functions. The pullback formula was originally proven for Siegel Eigenstein series by Garrett [Gar84]. To be precise this pullback formula is a way to write a Siegel Eisenstein series in terms of Siegel modular forms of lower genus. In this paper we include an exposition on Garrett's proof in the case of genus two Siegel Eisenstein series on Sp_4 restricted to $\mathbb{H} \times \mathbb{H} \subseteq \mathbb{H}_2$ in terms of classical modular forms on z and w by using a natural embedding of $SL_2 \times SL_2 \subseteq Sp_4$.

Ichino was able to generalize the above result to create a pullback formula for all Saito Kurokawa lifts, which include the above Siegel Eisenstein series [Ich05]. The pullback formula in this case has that for a genus two Saito-Kurokawa lift of weight k, F, and an eigenform basis of the space of genus one weight k modular forms $\{f_l\}$ we have

$$F\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = \sum_{l=1}^{n} c_l f_l(z) f_l(w)$$

with z, w lying in the complex upper half plane. The purpose of this paper is to find the above c_l values due to the following equality from Ichino's paper

$$\Lambda(2k-2, Sym^2(f_l) \otimes f) = 2^k \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathbb{H}\times\mathbb{H}}, f_l \times f_l\rangle|^2}{\langle f_l, f_l\rangle^2}$$

where f is a given modular form, h is associated to it by the Shimura correspondence and F is the corresponding Saito-KuroKawa lift of h, having weight k. Each f_l is a classical modular form of weight k, and $< \cdot, \cdot >$ is the petersson inner product defined on the space of modular forms. In this formula $\frac{|\langle F|_{\mathbb{H}\times\mathbb{H}}, f_l \times f_l\rangle|}{\langle f_l, f_l\rangle^2} = c_l$ from the previous formula. Thus using the pullback formula to find these c_l values we can then square them and multiply by a normalizing factor to find specific value of this symmetric square L-Function.

To find these c_l values I wrote a script in the programming language sage that was able to read in files from the modular form database LMFDB.org and output the desired values. This paper will include background on Modular forms, Hecke Operators, and L-functions, a write up of specific case Paul Garret's pullabck formula, Ichino's generalization, the work and code that I did to find these values, some of these found values, and future plans.

2 Background

2.1 Modular Forms

A modular form of weight k is a complex function that is holomorphic on the upper half plane $\mathbb{H}$ and holomorphic as $z \to i\infty$, that satisfies the property that

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

It is clear to see that the space of weight k modular forms, M_k, is a vector space but it is nontrivial that these vector spaces are finite dimensional[SS03]. This is an incredibly useful property for a variety of reasons but for this project it is primarily necessary for the computability of the coefficients in the pullback formula.

Siegel modular forms are a generalization of modular forms. The Saito-Kurokawa lifts we look at in this paper are Siegel Modular forms. We define the Siegel upper half space to be

$$\mathbb{H}_n = \{Z = X + iY \in M_n(\mathbb{C}) | X, Y \text{ real}, Z = {}^tZ, Y = (im)(Z) > 0\}.$$

Then a function $F : \mathbb{H}_n \to \mathbb{C}$ is a Siegel modular form of weight k and degree or genus n if F is holomorphic and

$$F((AZ+B)(CZ+D)^{-1}) = det(CZ+D)^k F(Z) \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}).$$

Where

$$Sp_n(\mathbb{Z}) = \{x \in M_{2n}(\mathbb{Z}) | \, {}^txJx = J\}$$

for $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Note that $Sp_1(\mathbb{F}) = SL_2(\mathbb{F})$. Also note that we often write $\mu(g, z) = det(cz+d)^{-2}$ for $g \in Sp_n(\mathbb{Z})$ which is called the factor of automorphy. The third criteria for being holomorphic at $i\infty$ that classical modular forms comes for free for all Siegel modular forms of genus two or higher by the Koecher principle [Koh].

For both modular forms and Siegel modular forms it is often important to look at their Fourier expansion. It is clear that modular forms have a Fourier expansion as $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in Sp_n(\mathbb{Z})$ gives the relation $f(z+1) = f(z)$. This is the form that these modular forms are read in from the database and the easiest to work with. These are given by

$$f(Z) = \sum_{\substack{T = {}^tT \; halfintegral \\ T \geq 0}} a(T) e^{2\pi i tr(TZ)}. \quad (2.1)$$

Additionally a modular form is called a cusp form if $T > 0$ in the above summation and the space of wight k cusp forms is denoted S_k.

It is also easy to show the equality $a({}^tUTU) = det(U)^k a(T) \quad \forall U \in GL_n(\mathbb{Z})$. Restricting U to $SL_n(\mathbb{Z})$ gives us $a({}^tUTU) = a(T)$. This will be useful later in our computations.

2.2 Hecke Operators

We define a Hecke operator $T_n : M_k \to M_k$ as

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

If we look at the Fourier expansion of a modular form then we can write

$$(T_n f)(z) = \sum_{m=0}^{\infty} b_n(m) q^m$$

where it is convention to write $q = e^{2\pi iz}$ and $b_n(m) = \sum_{d|(n,m)} d^{k-1} a(\frac{mn}{d^2})$ and $a(m)$ are the Fourier coefficients of f. [Haz]

We can define an inner product called the Petersson inner product on the vector space of modular forms: $\langle f, g\rangle := \int_{\mathrm{D}} f(\tau)\overline{g(\tau)}(\operatorname{Im}\tau)^k d\nu(\tau)$ where D is a fundamental domain and $d\nu(\tau)$ is the hyperbolic volume form. Under this inner product the Hecke operators are self adjoint. Because of this and fact that these operators commute we can find a basis that consists of modular forms that are simultaneous eigenvectors of all Hecke operators. This is called an eigenform basis. A given eigenform is normalized if $a(1) = 1$ in its Fourier expansion.

2.3 Standard L-Functions

For a given modular form, f, with Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ we can define its L-function by writing

$$L(s, f) = \sum_{n=0}^{\infty} a(n)n^{-s}$$

for $s \in \mathbb{C}$ a complex variable. For $f \in S_k$, $L(s, f)$ converges absolutely for all s with $\mathrm{Re}(s) > k/2 + 1$. If f is not a cusp form then $L(s, f)$ converges absolutely for all s with $\mathrm{Re}(s) > k$ [DS05].

If f is a Hecke Eigenform with weight k then $L(s, f)$ has an Euler product. Letting $T_p f = \lambda_p f$, we have

$$L(s, f) = \prod_p (1 - \lambda_p p^{-s} + p^{k-1-2s})^{-1}$$

where the product is taken over all primes. We also know that $a(p) = \lambda_p$. [DS05] Writing $X = p^{-s}$ we can see

$$1 - \lambda_p p^{-1} + p^{k-1-2s} = 1 - \lambda_p X + p^{k-1} X^2 = (1 - \alpha_p X)(1 - \beta_p X)$$

for $\alpha_p + \beta_p = \lambda_p$ and $\alpha_p \beta_p = p^{k-1}$. So

$$L(s, f) = \sum_{n=0}^{\infty} a(n) n^{-s} = \prod_p (1 - \alpha_p X)^{-1} (1 - \beta_p X)^{-1}$$

2.4 Symmetric Square L-function

We can define the symmetric square L-function for a modular form, f, as

$$L(f, sym^2, s) = \prod_p (1 - \alpha_p^2 X)^{-1} (1 - \alpha_p \beta_p X)^{-1} (1 - \beta_p^2 X)^{-1}.$$

We want to write the above in terms of the Fourier coefficients of f using what we know about the standard L-Function. First, for a normalized eigenform, f, it Fourier coefficients have the relation

$$a(p^r) = a(p) a(p^{r-1}) - p^{k-1} a(p^{r-2})$$

so, as we know that $a(1) = 1$ and $a(p) = \lambda_p = \alpha_p + \beta_p$ we can use induction to show

$$a(p^r) = \alpha_p^r + \alpha_p^{r-1} \beta_p + \cdots + \alpha_p \beta_p^{r-1} + \beta_p^r.$$

It is easy to see that $a(p^1) = \alpha_p + \beta_p$ and

$$a(p^2) = (\alpha_p + \beta_p)(\alpha_p + \beta_p) - \alpha_p \beta_p = \alpha_p^2 + \alpha_p \beta_p + \beta_p^2$$

and then

$$\begin{aligned} a(p^r) &= (\alpha_p + \beta_p)(\alpha_p^{r-1} + \alpha_p^{r-2} \beta_p + \cdots + \alpha_p \beta_p^{r-2} + \beta_p^{r-1}) - \alpha_p \beta_p (\alpha_p^{r-2} + \alpha_p^{r-3} \beta_p + \cdots + \alpha_p \beta_p^{r-3} + \beta_p^{r-2}) \\ &= \alpha_p^r + \alpha_p^{r-1} \beta_p + \cdots + \alpha_p \beta_p^{r-1} + \beta_p^r \end{aligned}$$

as desired. And this is equal to $\frac{\alpha_p^{r+1}-\beta_p^{r+1}}{\alpha_p-\beta_p}$ So, we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} a(p^{2k})X^{2k} &= \sum_{k=0}^{\infty} \frac{\alpha_p^{2k+1}-\beta_p^{2k+1}}{\alpha_p-\beta_p} X^{2k} \\
&= \frac{1}{\alpha_p-\beta_p}(\alpha_p(\sum_{k=0}^{\infty}\alpha_p^{2k}X^{2k}) - \beta_p(\sum_{k=0}^{\infty}\beta_p^{2k}X^{2k})) \\
&= \frac{1}{\alpha_p-\beta_p}(\frac{\alpha_p}{1-\alpha_p^2X^2} - \frac{\beta_p}{1-\beta_p^2X^2}) \\
&= \frac{\alpha_p - \alpha_p\beta_p^2X^2 - \beta_p + \alpha_p^2\beta_pX^2}{(\alpha_p-\beta_p)(1-\alpha_p^2X^2)(1-\beta_p^2X^2)} \\
&= \frac{1+\alpha_p\beta_pX^2}{(1-\alpha_p^2X^2)(1-\beta_p^2X^2)} \\
&= \frac{1-\alpha_p^2\beta_p^2X^4}{(1-\alpha_p^2X^2)(1-\alpha_p\beta_pX^2)(1-\beta_p^2X^2)} \\
&= \frac{1-p^{2k-2-4s}}{(1-\alpha_p^2X^2)(1-\alpha_p\beta_pX^2)(1-\beta_p^2X^2)}
\end{aligned}$$

We can then take product of the above over all primes p to get

$$\frac{1}{\zeta(2k-2-4s)}L(f,sym^2,s) = \prod_p \sum_{k=0}^{\infty} a(p^{2k})p^{-2s} = \sum_{n=0}^{\infty}\frac{a(n^2)}{n^s}$$

This alternative definition of the symmetric square L-function give an idea of why it appear in the pullback formula. In the proof of Garret's pullback we will see a sum over the square of all integers in the symmetric square operator defined below.

2.5 Garrett's Pullback Formula

The following is a rewrite Paul Garrett's Proof of the pullback of Eisenstein series [Gar84] in less generality, in the case of level 1, genus 2 Eisenstein series.

We will be computing the pullback of a Siegel's Eisenstein series via the maps:

$$\mathbb{H}\times\mathbb{H} \ni (z,w) \hookrightarrow \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \in \mathbb{H}_2$$

$$SL_2\times SL_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \hookrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & c' \end{pmatrix} \in Sp_2.$$

The explicit theorem is as follows:

Theorem 2.2. *Let $E_k^{(2)}$ be the Siegel's Eisenstein series of weight k on $\mathbb{H}_2$. Then for $2k > 3$,*

$$E_k^{(2)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = E_k^{(1)}(z)E_k^{(1)}(w) + {c_k^{(1)}}^{-1} \sum_{j=1}^{dim(S_k)} S_j f_j(z) f_j^{\theta}(w)$$

where $\{f_i : i\}$ is an orthonormal basis for cuspforms on $\mathbb{H}$, consisting of eigenvectors for the symmetric square operator (see §4), with eigenvalues $\{S_i : i\}$. And the θ-operator complex conjugates the Fourier coefficients on the f_i's

The only real calculation to worked out is the determination of coset representative for

$$\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Sp_2(\mathbb{Z}) \setminus Sp_2(\mathbb{Z}) \ / \ SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \right\}$$

which is worked out in §2 (and related coset computations are in §3).

2.5.1 Additional Background

The Siegel's Eisenstein series of weight k (of $\mathbb{H}_n$) is

$$E_k^{(n)}(z) = \sum_{\{c,d\}} det(cz+d)^{-2k}$$

where $\{c, d\}$ indicates that the sum is over (n-by-n) symmetric coprime pairs (c,d) left modulo $GL_n(\mathbb{Z})$. Which is equivalent to summing over pairs (c,d) for which there exist (a,b) such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n(\mathbb{Z})$. This series is convergent for $2k > n+1$

Now, define the "standard" maximal parabolics of Sp_n for $0 \leq r \leq n$:

$$P_{n,r} = Z_{n,r} G_{n,r}$$

where $Z_{n,r}$ is the subgroup of Sp_n with elements of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & 1_r & * & 0_r \\ 0 & 0 & * & 0 \\ 0 & 0 & * & 1_r \end{pmatrix}$$

and $G_{n,r}$ consists of elements of Sp_n of the form

$$\begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & * & 0 & * \end{pmatrix}$$

Note that $G_{n,r} \cong Sp_r$.

The reproducing kernel $K_k^{(n)}$ for the space of weight-k cuspforms on $\mathbb{H}_n$ is

$$K_k^{(n)}(z,w) = c_k^{(n)} \sum_{g \in Sp_n(\mathbb{Z})} \mu(g,z)^k det(g(Z) - \overline{w})^{-2k}$$

with some constant $c_k^{(n)}$. For a cuspform f, we have

$$f(z) = \int_{Sp_n(\mathbb{Z})\backslash\mathbb{H}_n} f(w) K_k^{(n)}(z,w)(det(Im(w)))^k dw$$

where dw is the $Sp_n(\mathbb{R})$-invariant measure on $\mathbb{H}_n$. Equivalently,

$$K_k^{(n)}(z,w) = \sum_{j=1}^{d(n)} f_j(z)\overline{f_j(w)}$$

where $\{f_j\}$ is an orthonormal basis for the space of cuspforms. [Kli90]

2.5.2 The Double Coset Decomposition

We consider $SL_2 \times SL_2$ imbedded in Sp_2 by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & c' \end{pmatrix}$$

and when convenient will identify these groups with their images.

Theorem 2.3. *$P_{2,0}(\mathbb{Z})\backslash Sp_2(\mathbb{Z})/SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ has an irredundant set of coset representatives*

$$g_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & M & 1 & 0 \\ M & 0 & 0 & 1 \end{pmatrix}$$

where $0 \neq M \in \mathbb{Z}$.

Proof. The proof depends on the fact that $\mathbb{Z}$ is a noetherian principal ideal domain. Over a field the computation is much simpler.

It is well-known that

$$P_{2,0}(\mathbb{Z})\backslash Sp_2(\mathbb{Z})$$

is in one-to-one correspondence with

$$GL_2(\mathbb{Z})\backslash\{\text{symmetric coprime pairs of size 2-by-2}\}.$$

Let $SC(n)$ be the set of n-by-n symmetric coprime pairs, and put

$$L_1 = \left\{ g \in Sp_1(\mathbb{Z}) | g = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}.$$

Lemma 2.4. *Every coset in $GL_2(\mathbb{Z})\backslash SC(2)/SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ contains an element of the form*

$$\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} c'_{11}d'_{11} & c'_{12} & d'_{11} & d'_{12} \\ c'_{21}d'_{11} & c'_{22} & 0 & d'_{22} \end{pmatrix}.$$

Proof. Letting $d'_{11} = \gcd(c_{11}, d_{11}, c_{21}, d_{21})$, we know by Bezout's identity and by using a modified smith normal form algorithm, that we can put it in the correct location by acting on the left by $GL_2(\mathbb{Z})$ and right by $SL_2(\mathbb{Z})$. We can then eliminated the entry below it by an operation of $GL_2(\mathbb{Z})$ on the left.

□

Lemma 2.5. *The double coset representation of the previous lemma may be further normalized to the form*

$$\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} c'_{11}d'_{11} & c''_{12} & d'_{11} & d''_{12} \\ c'_{21}d'_{11} & 0 & 0 & d''_{22} \end{pmatrix}.$$

Proof. Take

$$\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} c'_{11}d'_{11} & c'_{12} & d'_{11} & d'_{12} \\ c'_{21}d'_{11} & c'_{22} & 0 & d'_{22} \end{pmatrix}$$

as in the previous lemma. We can find a $g \in SL_2(\mathbb{Z})$ so that

$$(c'_{22}\ d'_{22})g = (0\ d''_{22}).$$

Thus,

$$\begin{pmatrix} c & d \end{pmatrix} g = \begin{pmatrix} c'_{11}d'_{11} & c''_{12} & d'_{11} & d''_{12} \\ c'_{21}d'_{11} & 0 & 0 & d''_{22} \end{pmatrix}.$$

□

Lemma 2.6. *In the special coset representative shown as in the previous lemma, $d''_{22} = 1$.*

Proof. By the "coprimeness" of $(c\ d)$, each row has gcd 1. That is:

$$c'_{21}d'_{11}x + d''_{22}y = 1.$$

But by the symmetry of

$$\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} c'_{11}d'_{11} & c''_{12} & d'_{11} & d''_{12} \\ c'_{21}d'_{11} & 0 & 0 & d''_{22} \end{pmatrix},$$

$$\begin{pmatrix} c'_{11}d'_{11} & c'_{12} \\ c'_{21}d'_{11} & 0 \end{pmatrix}^t \begin{pmatrix} d'_{11} & d''_{12} \\ 0 & d''_{22} \end{pmatrix} = \begin{pmatrix} d'_{11} & d''_{12} \\ 0 & d''_{22} \end{pmatrix}^t \begin{pmatrix} c'_{11}d'_{11} & c''_{12} \\ c'_{21}d'_{11} & 0 \end{pmatrix},$$

$c'_{21}d'^2_{11} = d''_{22}c''_{12}$, so $d_{22''} = 1$ □

Lemma 2.7. *We can further normalize the representative of the previous lemma to the form*

$$\begin{pmatrix} c''_{11}d'_{11} & d'^2_{11}c'_{21} & d'_{11} & 0 \\ c'_{21}d'_{11} & 0 & 0 & 1 \end{pmatrix}.$$

Proof. Starting with the form of the previous lemma, left multiply by

$$\begin{pmatrix} 1 & -d''_{12} \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$$

to anahilate the (1,4) entry. Then the symmetry gives the (1,2) block. □

Lemma 2.8. *In the noramlized form of the previous lemma, $d'_{11} = 1$.*

Proof. By coprimeness the top row has gcd=1 but clearly each element is divisible by d'_{11}, thus $d'_{11} = 1$.

□

Now we can finish the proof by first multiplying on the right by

$$\begin{pmatrix} 1 & 0 \\ -c''_{11} & 1 \end{pmatrix}$$

to obtain

$$\begin{pmatrix} 0 & c'_{21} & 1 & 0 \\ c'_{21} & 0 & 0 & 1 \end{pmatrix}.$$

To prove uniqueness, suppose that $g \in GL_2(\mathbb{Z}), g' \in SL_2(\mathbb{Z}), g'' \in SL_2(\mathbb{Z})$, and M and M' are such that

$$g \begin{pmatrix} 0 & M & 1 & 0 \\ M & 0 & 0 & 1 \end{pmatrix} g'g'' = \begin{pmatrix} 0 & M' & 1 & 0 \\ M' & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$g \begin{pmatrix} 0 & 1 \\ M & 0 \end{pmatrix} g' = \begin{pmatrix} 0 & 1 \\ M' & 0 \end{pmatrix},$$

$$g \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} g'' = \begin{pmatrix} M' & 0 \\ 0 & 1 \end{pmatrix}.$$

By uniqueness of elementary divisor form, $M = M'$.

So, given such M, clearly

$$g_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & M & 1 & 0 \\ M & 0 & 0 & 1 \end{pmatrix}$$

is in $Sp_2(\mathbb{Z})$. This proves the theorem.

□

2.5.3 The Twisted Coset Decomposition

Theorem 2.9. *With g_M as above, $M \neq 0$, and* $P_{2,0} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$,

$$P_{2,0} \backslash P_{2,0} g_M SL_2(\mathbb{Z}) SL_2(\mathbb{Z})$$

has an irredundant set of coset representatives

$$g_M g_0' g_1''$$

where

$$g_0' \in SL_2(\mathbb{Z})$$

and

$$g_1'' \in \Gamma(M) \backslash SL_2(\mathbb{Z})$$

where $g_0' g_1'' \in Sp_2(\mathbb{Z})$ as before and $\Gamma(M)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ of elements g so that

$$\begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} g \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Proof. It suffices to show that

$$P_{2.0}(\mathbb{Z}) g_M g' g'' = P_{2,0}(\mathbb{Z}) g_M$$

iff

$$g', g'' \in SL_2(\mathbb{Z})$$

and

$$g'' = \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} g' \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix}$$

identifying g' and g'' with $SL_2(\mathbb{Z})SL_2(\mathbb{Z}) \subseteq Sp_2(\mathbb{Z})$. We look at the condition

$$g_M g' g'' g_M^{-1} \in P_{2,0}(\mathbb{Z}).$$

This follows by multiplying-out. Put

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, g'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$$

then

$$g_M g' g'' g_M^{-1} = \begin{pmatrix} a' & -Mb' & b' & 0 \\ -Mb'' & a'' & 0 & b'' \\ c' - M^2 b'' & M(a'' - d') & d' & Mb'' \\ M(a' - d'') & c'' - M^2 b' & Mb' & d'' \end{pmatrix}$$

which is in $P_{2,0}$ when $c' - M^2 b'' = M(a'' - d') = M(a' - d'') = c'' - M^2 b' = 0$ which is ensured by the condition that

$$g'' = \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} g' \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix}.$$

□

2.5.4 The Symmetric-Square Operator

For a Siegel modular form f of weight k on $\mathbb{H}$, $M \in \mathbb{Z}_{>0}$, define

$$(T_M f)(z) = \sum_g f(M^2 g(z)) \mu(g, z)^k,$$

where g is summed over $\Gamma_n(M) \backslash SL_2(\mathbb{Z})$, with $\Gamma_n(M)$ as in the Theorem of §3 . We define the $\underline{symmetric - square}$ operator $S_n = S_n^{(k)}$ by

$$S_n = \sum_M T_m$$

where $M \in \mathbb{Z}_{>0}$ as above.

Proposition 2.10. *(Assuming the convergence of the series $S_n f$ for a cuspform f) $S_n = S_n^{(k)}$ is a hermitian operator on the space of weight-k cuspforms on $\mathbb{H}_n$, with respect to the Petersson inner product. Further, the eigenspaces of S_n are spanned by cuspforms with algebraic coefficients.*

Proof. First, we can see that

$$(det M)^{2k} T_M = T'_M$$

where T'_M is

$$(T'_M f)(z) = \sum_g f(g(z))\mu(g,z)^k,$$

where g is summed over

$$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{Z})\begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix} SL_2(\mathbb{Z}).$$

Then one can check by methods described in chapter 3 of [Shi71] that T'_M is hermitian. Hence S_n is a sum of hermitian operators, so it is hermitian if the series converges.

For the second assertion recall that the space of cuspforms has a finite basis of cuspforms with rational Fourier coefficients. The operators T_M or T'_M map cuspforms with algebraic Fourier coefficients to cuspforms with algebraic Fourier coefficients. If we can show all the T'_Ms commute, then the second assertion would follow by linear algebra.

By the criterion of [Shi71], ch. 3 prop 3.8, if we can find an anti-involution * on $SL_2(\mathbb{Q})$ so that

$$\left(SL_2(\mathbb{Z})\begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix} SL_2(\mathbb{Z})\right)^*$$

$$= SL_2(\mathbb{Z})\begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix} SL_2(\mathbb{Z}),$$

then we have commutivity. It is easy to check that

$$g^* = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} g^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

works. This proves the proposition □

2.5.5 The main Formula

Theorem 2.11. *Let $E_k^{(2)}$ be the Siegel's Eisenstein series of weight k on $\mathbb{H}_2$. Then for $2k > 3$,*

$$E_k^{(2)}\begin{pmatrix} z & 0 \\ 0 & w\end{pmatrix} = E_k^{(1)}(z)E_k^{(1)}(w) + {c_k^{(1)}}^{-1}\sum_{j=1}^{dim(S_k)} S_j f_j(z) f_j^{\theta}(w)$$

where $\{f_i : i\}$ is an orthonormal basis for cuspforms on $\mathbb{H}$, consisting of eigenvectors for the symmetric square operator (see §4), with eigenvalues $\{S_i : i\}$. Finally, the θ-operator complex conjugates the Fourier coefficients on the f_i's

Proof. This amounts to putting together our previous material, especially the coset decomposition of §2, §3, and the "cocylce formula" for μ that $\mu(g_1 g_2, z) = \mu(g_1, g_2(z))\mu(g_2, z)$.

First, for $M_0 = 1$,

$$\mu(g_{M_0}, \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) = det(\begin{pmatrix} 1 & w \\ z & 1 \end{pmatrix})^{-2} = (1 - zw)^{-2}.$$

Then

$$\begin{aligned} &\sum_{g \in SL_2(\mathbb{Z})} \mu(g_{M_0} g, \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix})^k \\ &= \sum_{g \in SL_2(\mathbb{Z})} \mu(g, z)^k (1 - g(zw))^{-2k} \\ &= \sum_{g \in SL_2(\mathbb{Z})} \mu(g, z)^k (gz - gw)^{-2k}, \end{aligned}$$

since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \xi \mapsto -\xi^{-1}$$

is in $SL_2(\mathbb{Z})$. But this expression is just

$$c_k^{(1)^{-1}} K_k^{(1)}(z, -\overline{w})$$

where $K_k^{(r)}$ is the kernel function of §1, and $c_k^{(1)}$ is the constant there. This is, then,

$$c_k^{(1)^{-1}} \sum_{j=1}^{dim(S_k)} f_j(z) \overline{f_j(-\overline{w}))}$$

$$c_k^{(1)^{-1}} \sum_{j=1}^{dim(S_k)} f_j(z) f_j^{\theta}(w))$$

where θ is as in the statement of the Theorem, and $\{f_j : j\}$ is any orthonormal basis for cuspforms on $\mathbb{H}$.

For fixed M one similarly computes that

$$\begin{aligned} &\sum_{g'_0, g''_1} \mu(g_m g'_0 g''_1, \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix})^k \\ &= \sum_{g''} \mu(g'', w) c_k^{(1)^{-1}} K_k^{(1)}(z, -M^2 \overline{g'' w}) \end{aligned}$$

where g'_0, g''_1 are summed as in §3, and $g'' \in \Gamma(M) \backslash SL_2(\mathbb{Z})$. Then by the previousm this is

$$c_k^{(1)^{-1}} \sum_{j=1}^{dim(S_k)} f_j(z) (T_M f_j)^{\theta}(w).$$

Now by the Proposition of §4, we make take $\{f_j\}$ to be eigenvectors for S_1, as S_1 is hermitian. Then the above becomes, when summed over all M,

$$c_k^{(r)^{-1}} \sum_j S_j f_j(z) f_j^{\theta}(w).$$

This gives the last term asserted formula. For the first term we will omit how the identity coset ($M = 0$ in theorem 2.3) gives rise to the Eisenstein Series. This can be found in the full proof by Garret. With regards to convergence, we see that all series mentioned are subseries of $E_k^{(2)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$, so are absolutely convergent uniformly for (z, w) in compact subsets of $\mathbb{H}_2$. Thus the theorem is proven.

□

3 Generalizations beyond Eisenstein Series

3.1 Saito-Kurokawa lifts

Let $f \in S_{2k-2}(\Gamma_1)$ be a cusp form of genus one and weight $2k-2$, with k even. Kohnen [Koh80] gave a one to one correspondence between the space $S_{2k-2}(\Gamma_1)$ and the space $S^+_{k-1/2}(\Gamma_0(4))$. The latter space is the space of cusp forms of weight $k - 1/2$ with respect to $\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : 4|c \right\}$, whose nth Fourier coefficient vanishes whenever $(-1)^{k-1}n \equiv 2, 3 \pmod 4$. Let $g \in S^+_{k-1/2}(\Gamma_0(4))$ correspond to f, and let g have Fourier coefficients $\{b(n)\}$. For a posisitve symmetric, and half integral 2×2 matrix T, define

$$a(T) = \sum_{d|gcd(T)} b(\frac{\det(2T)}{d^2})d^{k-1}.$$

Then we can define $F : \mathbb{H}_2 \to \mathbb{C}$ by the Fourier expansion $F(Z) = \sum_T a(T)e^{2\pi i tr(TZ)}$. Then $F \in S_k(\Gamma_2)$ and $F \neq 0$ if $f \neq 0$ [Pit19].

3.2 Ichino's Paper

In his paper on Saito-Kurokawa lifts, Ichino primarily proved the algebraicity of these values of L-functions. Specifically, let κ be an odd positive integer, $f \in S_{2k}(SL_2(\mathbb{Z}))$ be a normalized hecke eigenform, and $h \in S^+_{\kappa+1/2}(\Gamma_0(4))$ be a Hecke eigenform associated to f by the Shimura correspondence. Let $F \in S_{k+1}(Sp_2(\mathbb{Z})$ be the Saito-Kurokawa lift of h. For each normalized Hecke eigenform $g \in S_{\kappa+1}(SL_2(\mathbb{Z})$ we consider the period integral $\langle F|_{\mathbb{H}\times\mathbb{H}}, g \times g\rangle$.

Let $\Lambda(s, Sym^2(g) \otimes f)$ be the completed L-function given by

$$\Lambda(s, Sym^2(g) \otimes f) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-\kappa)\Gamma_{\mathbb{C}}(s - 2\kappa + 1)L(s, Sym^2(g) \otimes f) \tag{3.1}$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. It satisfies the functional equation:

$$\Lambda(4\kappa - s, Sym^2(g) \otimes f) = \Lambda(s, Sym^2(g) \otimes f). \tag{3.2}$$

The main result of this paper is that

$$\Lambda(2\kappa, Sym^2(g) \otimes f) = 2^{\kappa+1} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathbb{H}\times\mathbb{H}}, g \times g\rangle|^2}{\langle g, g\rangle^2}. \tag{3.3}$$

The term $\frac{|\langle F|_{\mathbb{H}\times\mathbb{H}}, g\times g\rangle|}{\langle g,g\rangle^2}$ is the coefficient in the pullback formula c_l which we are trying to find. This can be seen because taking the inner product of our saito-kurokawa lift restricted to $\mathbb{H} \times \mathbb{H}$ with a eigenform basis vector makes all other basis vectors drop out and leaves our c_l coefficient. This means that finding these c_l values gives us a way to compute these specific values of this normalized symmetric square L function.

4 Pull-Back Formula

While we proved Garrett's pullback formula above, we will use the generalized pullback formula that states for a Saito-Kurokawa, F, can be restricted to $\mathbb{H} \times \mathbb{H}$ and written in terms of an eigenform basis of genus one cusp forms $\{f_l\}_{l=1}^{n}$

$$F\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = \sum_{l=1}^{n} c_l f_l(z) f_l(w) \tag{4.1}$$

Where the c_ls are the values we are trying to which are equal to the last term in (3.3) Letting $a_F(T)$ be the Fourier coefficient of F corresponding to half integral matrix positive matrix T and $a_l(n)$ be the nth Fourier coeffienct of the lth basis vector of the eigenform basis of cusp forms. Then the above formula gives us the relation

$$\sum_{T=\begin{pmatrix} n & * \\ * & m \end{pmatrix}} a_F(T) = \sum_{l=1}^{n} c_l a_l(n) a_l(m). \tag{4.2}$$

5 The Code for this Computation

To find the c_l values mentioned above I wrote a program in the sage programming language. This code takes in given modular forms and outputs all c_l values for the given weight pullback.

The first step in doing this computation was downloading the Fourier expansions of the Saito-Kurokawa lifts and eigenform basis of modular forms from LMFDB.org. The Saito-Kurokawa lifts were saved in json files and easily readable, while the genus one modular forms were downloaded in a short program to generate the expansion. Both were given in the form of an indeterminate of a field extension which was specified. It is conjectured by

Maeda and believed to be true that all of the eigenform basis vectors lie in the same Galois orbit.

This meant it was necessary to find a common field to work in. We defined K to be a field in $'a'$ to be the field containing the coefficients of the Saito-Kurokawa lifts and L to be a field in $'b'$ that contains the coefficients of the eigenform basis of modular forms. Because all of the basis modular forms are in the same Galois orbit and because we need to differentiate them to form a linear system of equations we define O to be the Galois closure of L so we can compute the Galois conjugates of the given basis vector in O. Lastly, the field we work in is F, the composite field of K and O.

Because we are going to be solving a linear system of equations we next define a $1 \times n$ matrix where n is the dimension of L and a vector. Next we have outer loops over the diagonal entries of a 2×2 matrix. Within this loop we compute the left and right hand sides of equation 3.2 for the given diagonal entries.

The right hand side is simple to compute. If i and j are the diagonal entries from the outer loop then we multiply the ith and jth Fourier coefficients of the provided eigenform basis. We then find the galois conjugates of this value in F and append a row vector with these values to the above matrix.

Looking at the left hand side, there are finite T that are half integral and positive definite with a given diagonal. We iterate over each of them and look up the corresponding Fourier coefficient. Because matrices in the same $GL_2(\mathbb{Z})$ orbit under the action of ${}^tgTg$ with $g \in GL_2(\mathbb{Z})$ have the same Fourier coefficient, the data file only contains a representative of each orbit. So we use the built in command to test equivalence of binary quadratic forms to compare a given T to each representative in the datafile with the same determinant until the representative corresponding to T is found and the correct coefficient is found. The coefficients are summed across all T with the same diagonal and then put into the field F and appended to our vector we created earlier.

At the end of this iteration of the outer loops we compute the rank of the generated matrix. If it is not rank n then we continue onto a new set of diagonal entries to the matrix. If it is rank n then we break out of the loop and solve the system of equations corresponding to the matrix with solution of the given vector. By solving this we obtain the c_l's as desired. They are given as algebraic numbers in the field F.

6 Findings

Below is a table of the found c_l values for given small weights.

Weight	Field Polynomial	Computed values with field indeterminate 'a'
16	$x^2 - x - 12837$	$24a + 1524$
18	$x^2 - x - 589050$	12a - 6900
20	$x^2 - x - 15934380$	-84a - 258720
22	$x^3 - x^2 - 25398986824x - 1557240438880016$	$\frac{-437760}{91}a^2 + \frac{39329852400}{91}a + \frac{7492620289861440}{91}$

7 Conclusion and Future Plans

We have successfully found many of these c_l values but due to computational limitations it still takes significant amounts of time to find these values in modular forms of large weight. The dimension of these spaces of modular forms increases very quickly as the weight increases which corresponds to a larger field polynomial and a much longer time doing any computation. More time could be spent on simply optimizing the code as well as looking into running it on larger computing clusters. But we have successfully created a tractable program to compute these values which can easily be turned into values of the related symmetric square L functions.

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